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NEW RESULTS ON THE VIBRATING STRING WITH A CONTINUOUS OBSTACLE.(U)
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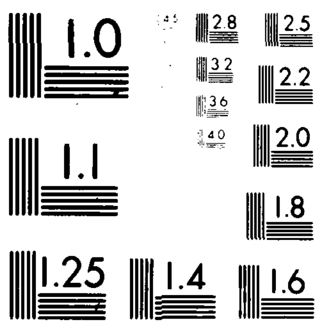

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NEW RESULTS ON THE VIBRATING STRING WITH A CONTINUOUS OBSTACLE

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NEW RESULTS ON THE VIBRATING STRING WITH A CONTINUOUS OBSTACLE.

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ABSTRACT

We give an explicit formula which describes the solution of the problem of the vibrating string with a plane obstacle. This formula allows us to prove continuous dependence on the data; a regularity result is given. We prove some results on the convergence of the penalized problem, and give a numerical scheme.

A few results are given without the requirement that the obstacle be plane.

AMS(MOS) Subject Classification: 35L15, 35L20, 35L67, 35L70, 65M05, 65M15, 65M25, 73D35.

Key Words: vibrating string; unilateral constraints; regularity; continuous dependence on data; penalty method; numerical scheme.

Work Unit No. 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The purpose of this paper is to continue the study of the following problem: a string is constrained to remain on one side of a material obstacle. We consider its transverse vibrations, and we assume that, when the string hits the obstacle, no energy is lost. It was proved in a previous paper that when the obstacle is concave i.e. has the form of a bowl (with the plane obstacle as a limiting case), then there exists a unique solution for a given initial position and velocity of the string. We prove here that when the initial position and velocity are slightly changed, the solution is only slightly changed. Moreover, if we replace the rigid obstacle by an elastic one, if the obstacle is plane, and if the initial data are sufficiently regular, then the solution we obtain is close to the solution of the original problem. We also give a numerical scheme for computing the solution.

This subject is a first approach to considering problems of mechanical vibrations with unilateral constraints; a number of elliptic and parabolic problems with unilateral constraints have been solved, but very few hyperbolic problems, i.e. problems describing vibrating modes such as the above obstacle problem, have been tackled successfully.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NEW RESULTS ON THE VIBRATING STRING WITH A CONTINUOUS OBSTACLE

A. Bamberger[‡]
M. Schatzman[†]

I. INTRODUCTION

I.1. Presentation of the problem and the results.

This paper aims to give some new results on vibrating strings with obstacles. The model is the same as in [1], but as it appears necessary to elucidate several points of the modelization which was exposed there, we shall give it from the beginning.

We consider the small transverse vibrations of a string that is constrained to be on one side of a material obstacle. Let the transverse displacement at time t of the material point of the string with coordinate x be denoted by $u(x,t)$. If the string were free, i.e. if there were no obstacle, then u would satisfy the wave equation

$$\Box u \equiv u_{tt} - u_{xx} = 0 .$$

We assume that the obstacle has position $\varphi(x)$. We translate the requirement that the string stay on one side of the obstacle into the inequality

$$(1) \qquad u(x,t) \geq \varphi(x) \qquad \forall \ x,t .$$

When the string does not touch the obstacle, its motion satisfies the wave equation, and thus

$$(2) \qquad \text{supp } \Box u \subset \{(x,t)/u(x,t) = \varphi(x)\}.$$

We require that the string does not stick to the obstacle; this can be translated as

$$(3) \qquad \Box u \geq 0 ,$$

which means that the obstacle does not exert a downwards force on the string.

Notice that (3) is essentially equivalent to subsonic propagation of interactions. To see this, let $t = \sigma(x)$ a curve which separates a region $\mathcal{N}$ of the half plane $\mathbb{R} \times (0,\infty)$ in two open regions $\mathcal{N}^+$ and $\mathcal{N}^-$ where $\Box u$ vanishes. Suppose that $u^+ = \ldots$

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and $u^- = u|_{\Omega^-}$ are sufficiently smooth, and that

(4) $$u^{\pm}(x,\sigma(x)) = \varphi(x)$$

(5) $$u^{\pm}(x,t) \geq \varphi(x) \qquad \forall\ (x,t) \in \Omega.$$

Then we can compute $\Box u$ in the sense of distributions, with ψ a test function:

(8) $$\left\{\begin{aligned} &\langle \Box u, \psi\rangle = -\left\langle \frac{\partial u}{\partial t}, \frac{\partial \psi}{\partial t}\right\rangle + \left\langle \frac{\partial u}{\partial x}, \frac{\partial \psi}{\partial x}\right\rangle = \\ &= \int \left[\left(\frac{\partial u^+}{\partial t} - \frac{\partial u^-}{\partial t}\right)(x,\sigma(x)) + \left(\frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x}\right)(x,\sigma(x))\,\sigma'(x)\right]\psi(x,\sigma(x))\,dx. \end{aligned}\right.$$

Relation (4) can be differentiated, with respect to x, and implies

(7) $$\left(\frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x}\right)(x,\sigma(x)) = -\sigma'(x)\left(\frac{\partial u^+}{\partial t} - \frac{\partial u^-}{\partial t}\right)(x,\sigma(x)) .$$

Introducing (7) into (8), we get:

$$\langle \Box u, \psi\rangle = \int \left(\frac{\partial u^+}{\partial t} - \frac{\partial u^-}{\partial t}\right)(x,\sigma(x))(1 - \sigma'^2(x))\,\psi(x,\sigma(x))\,dx.$$

But hypothesis (4) and (5) ensure that

$$\frac{\partial u^+}{\partial t}(x,\sigma(x)) \geq 0 \quad \text{and} \quad \frac{\partial u^-}{\partial t}(x,\sigma(x)) \leq 0 .$$

Therefore, $\Box u$ is non-negative if and only if $|\sigma'|$ is almost everywhere smaller than 1.

It is not enough to suppose that conditions (1), (2) and (3) are satisfied, as nothing has been said of the evolution of the energy of the string during the collision with the obstacle.

The hypothesis that will be made is that the energy is conserved. This requirement should be analysed from a mathematical point of view as follows: the condition must be local, because the propagation properties of hyperbolic equations suggest it, and it must be satisfied wherever in the x,t half-plane the free wave equation is satisfied. Thus, multiplying by $\frac{\partial u}{\partial t}$ the relation

(8) $$\Box u = 0 \quad \text{on} \quad \Omega,$$

where Ω is an open region such that (8) is satisfied, we obtain a relation in divergence form

(9) $$\frac{\partial}{\partial t}\left\{\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2\right\} - \frac{\partial}{\partial x}\left\{2\,\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial x}\right\} = 0 \quad \text{in} \quad \Omega .$$

The operations by which we deduce (9) out of (8) are valid if $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ are locally square-integrable in $\mathbb{R} \times (0,\infty)$.

The energy condition we shall impose is

$$(10)\qquad \begin{cases} \dfrac{\partial}{\partial t}\left(\left|\dfrac{\partial u}{\partial t}\right|^2 + \left|\dfrac{\partial u}{\partial x}\right|^2\right) - \dfrac{\partial}{\partial x}\left(2\,\dfrac{\partial u}{\partial t}\,\dfrac{\partial u}{\partial x}\right) = 0 \\ \\ \text{in the sense of distribution on } \mathbb{R}\times(0,\infty). \end{cases}$$

We could alternatively write it as

$$(11)\qquad \begin{cases} S_u \stackrel{\text{def}}{=} (-2u_x u_t,\ u_x^2 + u_t^2) \\ \nabla \cdot S_u = 0 . \end{cases}$$

Here, the first component of the vector field S_u is the energy density flux, and the second component of the vector field S_u is the energy density.

Notice that (10) <u>cannot</u> be deduced by multiplying (3) by $\frac{\partial u}{\partial t}$, as $\frac{\partial u}{\partial t}$ must be expected to be discontinuous on the support of $\Box u$.

For initial conditions such that the free solution corresponding to them is locally of bounded energy, it was proved in [1] that the Cauchy problem (1)-(3) and (11) possesses a unique solution if the function φ is convex.

The approach which led to condition (11) is essentially a mathematical one; from the mechanical point of view, one would like to know if (11) implies that the velocity of the string after collision is the opposite of the velocity of the string before collision. The answer is affirmative, but one has to give a meaning to

$$(12)\qquad \frac{\partial u}{\partial t}(x,t+0) = -\frac{\partial u}{\partial t}(x,t-0) \quad \text{if} \quad (x,t) \in \operatorname{supp} \Box u .$$

This was the purpose of part V of [1], where it was shown that if

$$(13)\qquad \begin{cases} \sigma \text{ is Lipschitz continuous on } \mathbb{R}, \text{ with Lipschitz} \\ \text{constant } 1, \text{ and } \sigma \geq 0 \text{ on } \mathbb{R} \end{cases}$$

$$(14)\qquad \int_{-a}^{a} (|u_x(x,t)|^2 + |u_t(x,t)|^2)\,dx \leq C(a,b) \quad \forall\, a > 0,\ \forall\, b > 0,\ \forall\, t \leq b$$

and if (3) is satisfied, then right- and left- derivatives can be defined almost everywhere on the non characteristic parts of the curve $t = \sigma(x)$.

Moreover, if (11) holds, then for all σ satisfying (13), we have:

$$(15)\qquad \left|\frac{\partial^+ u}{\partial t}(x,\sigma(x))\right| = \left|\frac{\partial^- u}{\partial t}(x,\sigma(x))\right| \quad \text{a.e. on } \{x/|\sigma'(x)| < 1\}.$$

We shall prove in paragraph II the following explicit formula in the case of the plane obstacle.

Let w be the free solution of the wave equation

$$\begin{cases} \Box w = 0 \\ w(x,0) = u_0(x), \\ w_t(x,0) = u_1(x). \end{cases}$$

Let the obstacle be $\varphi = 0$, and let the backward wave cone be

$$T^-_{x,t} \overset{\text{def}}{=} \{(x',t')/0 \leq t' \leq t - |x-x'|\}.$$

Let us denote by r^- the negative part of a number

$$r^- = \sup(-r,0) .$$

Then the solution of the problem (1) - (3) and (11) is given by

$$u(x,t) = w(x,t) + 2 \sup\{(w(x',t'))^-/(x',t') \in T^-_{x,t}\}.$$

This formula shortens considerably a previous proof [6] of continuous dependence on data, and is the key for the numerical scheme studied in paragraph III. We shall give in paragraph IV a regularity theorem in spaces of bounded variation, in the case of a general concave obstacle.

In paragrapn V, we shall consider the functions u_λ which solve the problem

$$(16) \qquad \begin{cases} \Box u_\lambda - \frac{1}{\lambda}(u_\lambda - \varphi)^- = 0 \\ u_\lambda(x,0) = u_0(x) \\ \frac{\partial u_\lambda}{\partial t}(x,0) = u_1(x) \end{cases}$$

In the first part of this paragraph, we shall prove a weak convergence result, which does not depend on the shape of φ or on the regularity of the initial data. The limit function will satisfy a set of energy inequalities instead of (11).

In the second part, we shall assume that the obstacle is plane, and that $\frac{du_0}{dx}$ and u_1 are locally of bounded variation. Then the solution of (16) converges strongly in $H^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$, and its limit is the unique solution of (1)-(3) and (11).

I.2. Notations and summary of previous results.

We shall use throughout this paper the following notations and definitions:

$$(17)\quad \begin{cases} V \text{ is the set of functions } u \text{ such that} \\ \int_{-a}^{a} (|u_x(x,t)|^2 + |u_t(x,r)|^2)dx \leq C(a,b) < +\infty \quad \forall a,b \quad \forall t < b \end{cases}$$

w is the free solution of the wave equation:

$$(18)\quad \begin{cases} \Box w = 0 \ , \\ w(x,0) = u_0(x) \\ w_t(x,0) = u_1(x) \ . \end{cases}$$

E is the set

$$E = \overline{\{(x,t)/w(x,t) < \varphi(x)\}}$$

I is the domain of influence defined by

$$(19)\quad I = \cup\{T^+_{x,t}/(x,t) \in E\}$$

where $T^+_{x,r}$ is the forward wave cone $\{(x',r')/t' \geq t + |x-x'|\}$, and its boundary is given by

$$(20)\quad \begin{cases} \partial I = \{(x,r)/t = \tau(x)\} \\ \text{where } \tau \text{ is Lipschitz continuous with Lipschitz constant 1 .} \\ \text{(see [1], Proposition II.3 for the proof of this claim).} \end{cases}$$

The backward wave cone $T^-_{x,t}$ is $\{(x',t')/0 \leq t' \leq t - |x-x'|\}$.

The characteristic coordinates ξ and η are given by

$$(21)\quad \xi = \frac{x+r}{\sqrt{2}} \qquad \eta = \frac{-x+r}{\sqrt{2}}$$

with the notation $\tilde{z}(\xi,\eta) = z(\frac{\xi-\eta}{\sqrt{2}}, \frac{\xi+\eta}{\sqrt{2}})$ for all functions of two variables x and t.

We shall call problem (P_∞) the following problem: given $u_0 \in H^1_{loc}(\mathbb{R})$, $u_1 \in L^2_{loc}(\mathbb{R})$ satisfying the compatibility condition

$$(22)\quad \begin{cases} u_0(x) \geq \varphi(x) \\ u_1(x) \geq 0 \quad \text{a.e. on } \{x|u_0(x) = \varphi(x)\} \end{cases}$$

find u in V such that

$$(1)\quad u \geq \varphi$$

$$(2)\quad \operatorname{supp} \Box u \subset \overline{\{(u_1 t)/u(x,t) = \varphi(x)\}}$$

$$(3)\quad \Box u \geq 0$$

(10) $\begin{cases} \frac{\partial}{\partial t}(u_x^2 + u_t^2) + \frac{\partial}{\partial x}(-2u_x u_t) = 0 \text{ in the sense} \\ \text{of distributions in } \mathbb{R} \times \mathbb{R}^+ \end{cases}$

(23) $\begin{cases} u(x,0) = u_0(x) \\ \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{cases}$

The precise statement of the results of existence and uniqueness in [1] is as follows:

Theorem 0. The Problem P_∞ possesses a unique solution u if φ'' is non-negative. Moreover, this solution u is the unique solution of the linear problem

(24) $\begin{cases} u \in V \\ \Box u\,|_{\{(x,t)/t \neq \tau(x)\}} = 0 \\ \frac{\partial u}{\partial t}(x,\tau(x)+0) = -\frac{\partial u}{\partial t}(x,\tau(x)-0) \quad \text{a.w. on } \{x/\tau(x) > 0 \,\&\, |\tau'(x)| < 1\}. \\ u(x,0) = u_0(x) \\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \end{cases}$

If μ is the measure defined by

(25) $\langle \mu,\psi \rangle = -2 \int w_t(x,\tau(x))(1 - \tau'(x)^2)\ \psi\ (x,\tau(x))dx$,

then the solution of (19) is given by

(26) $u = w + \mathcal{E} * \mu$

where $\mathcal{E}$ is the elementary solution of the wave equation with support in the positive light cone:

(27) $\begin{cases} \mathcal{E} = \frac{1}{2} \text{ on } \{(x,t)/t \geq |x|\} \\ \mathcal{E} = 0 \quad \text{elsewhere.} \end{cases}$

It will be useful to consider the problems $P_{x,t}$, which are just P_∞ restricted to the backward wave cone $T^-_{x,t}$, with initial data given on $[x - t, x + t]$. Clearly, u is a solution of P_∞ if and only if it is a solution of $P_{x,t}$ for all $x \in \mathbb{R}$, $t > 0$. The first result on the convergence of the penalty method for the string with an obstacle was proved by A. Bamberger [2].

An explicit formula for the string with a point obstacle was obtained by L. Amerio in [3] and by M. Schatzman in [4], with a different argument.

Continuous dependence on the data and convergence of the penalty method for the point obstacle are proved in [4]. See also the results of C. Citrini [5], when regularity assumptions are relaxed.

II. THE EXPLICIT FORMULA. CONTINUOUS DEPENDENCE ON THE INITIAL DATA

II.1. The explicit formula for the infinite string.

In the case of the zero obstacle (and more generally, the plane obstacle), the solution of (P_∞) can be expressed by an explicit formula. We denote by $r^- = \sup(r,0)$ the negative part of a number.

Theorem 1. The unique solution of P when $\varphi = 0$ is given by

$$u(x,t) = w(x,t) + 2 \sup_{(x',t')\in T^-_{x,t}} [w(x',t')]^- \tag{28}$$

Remark 2. If the obstacle is plane, i.e., if $\varphi(x) = \alpha x + \beta$, then (28) can be generalized:

$$u(x,t) = w(x,t) + 2 \sup_{(x',t')\in T^-_{x,t}} [w(x',t') - \varphi(x')]. \tag{29}$$

To deduce (29) from (28) it is enough to consider $u - \varphi$, and notice that $\Box\varphi = 0$. ■

The *proof* of Theorem 1 comes in several steps.

The first step is the following result:

Lemma 3. The set where $\sup\{[w(x',t')]^-/(x',t') \in T^-_{x,t}\}$ does not vanish is the interior of the domain of influence *I*.

Proof. If $w(x',t') < 0$ for some (x',t') in the backward cone $T^-_{x,t}$, then (x,t) belongs to the forward cone $T^+_{x',t'}$ the vertex of which is in the interior of E. Thus (x,t) is in the interior of I. Conversely, if (x,t) belongs to the interior of I, then there exists a point (x',t') in the interior of E such that (x,t) belongs to the interior of $T^+_{x',t'}$. We can choose this (x',t') such that $w(x',t')$ be strictly negative, because the set of (x',t') such that $w(x',t') < 0$ is dense in the interior of E. Therefore, $\sup_{(x',t')\in T^-_{x,t}} [w(x',t')]^- > 0$. ■

Let us define

$$k(x,t) = \inf\{w(x',t')/(x',t') \in T^-_{x,t}\}. \tag{30}$$

Then, thanks to Lemma 3, we have, if u is defined by (28)

$$\begin{cases} u(x,t) = w(x,t) & \text{for } t \le \tau(x), \\ u(x,t) = w(x,t) - 2k(x,t) & \text{for } t \ge \tau(x). \end{cases} \tag{31}$$

Lemma 4. Let u_0 and u_1 satisfy the compatibility conditions (22), and let I be nonempty. Then the function k satisfies

(32) $$\Box k = 0 \text{ in the interior of } I .$$

Proof. Let us extend w to the whole plane $\mathbb{R}\times\mathbb{R}$, by solving the (backward) wave equation

(33) $$\begin{cases} w(x,0) = u_0(x) , \\ w_t(x,0) = u_1(x) , \\ \Box w = 0 \qquad \text{for } t < 0 , \quad x \in \mathbb{R} . \end{cases}$$

The assumption that I is not empty implies that, on the line of influence:

$$w(x,\tau(x)) = 0 \qquad \text{if } |\tau'(x)| < 1$$

$$w_t(x, (x)) < 0 \qquad \text{a.e. on } \{x/|\tau'(x)| < 1\}.$$

We shall prove that $w(x,t) \geq 0$ for $t \leq \tau(x)$, by essentially the same argument as in [1], Theorem IV.2. For the convenience of the reader, let us sketch it here.

Let $U = \{x/w(x,\tau(x)) > \varphi(x)\} = \bigcup_i \,]a_i,b_i[$

where the open sets $]a_i,b_i[$ are the connected components of U. Then Lemma II.6 of [1] tells us that

(34) $$\tau(x) = \min(\tau(a_i) + x-a_i,\ \tau(b_i) + b_i - x), \qquad \forall\, x \in [a_i,b_i] .$$

Therefore if we set

(35) $$\begin{cases} \xi_i = \dfrac{a_i + \tau(a_i)}{\sqrt{2}} & \xi_i' = \dfrac{b_i + \tau(b_i)}{\sqrt{2}} \\ \eta_i = \dfrac{-a_i + \tau(a_i)}{\sqrt{2}} & \eta_i' = \dfrac{-b_i + \tau(b_i)}{\sqrt{2}} \end{cases}$$

the line of influence in characteristic coordinates is such that

(36) $$\begin{cases} Y(\xi) = \eta_i & \text{if } \xi \in [\xi_i,\xi_i') \\ Y(\xi) = [\eta_i', \eta_i] & \text{if } \xi = \xi_i' \end{cases}$$

if Y is the multivalued mapping defined by

$$\eta \in t(\xi) \iff \frac{\xi+\eta}{\sqrt{2}} = \sigma\left(\frac{\xi-\eta}{\sqrt{2}}\right) .$$

Let $w(\xi,\eta) = f(\xi) + g(\eta)$, where f and g are in $H^1_{loc}(\mathbb{R})$. From (35), we deduce $f(\xi) + g(\eta_i') \geq 0$ for $\xi_i \leq \xi \leq \xi_i'$ and from (34), $f(\xi) + g(\eta_i) \geq 0$ for $\xi_i \leq \xi \leq \xi_i'$. As we must have $f(\xi_i) + g(\eta_i') = 0 = f(\xi_i') + g(\eta_i') = 0$, by definition of U, ξ_i, ξ_i', η_i and η_i', then

(37) $$f(\xi_i) = f(\xi_i') \leq f(\xi) \quad \forall \xi \in [\xi_i,\xi_i'] .$$

Similarly,

(38) $$g(\eta_i) = g(\eta_i') \leq g(\eta) \quad \forall \eta \in [\eta_i',\eta_i] .$$

On C, the complement of the set $\bigcup_i [\xi_i,\xi_i']$, we have

(39) $$f'(\xi) \leq 0 \quad \text{a.e.} .$$

The simplest way to see this is to notice that Y is one valued on C, and that

$$f(\xi) + g(Y(\xi)) = 0 \quad \text{on} \quad C$$

$$f(\xi') + g(Y(\xi)) \geq 0 \quad \text{for} \quad \xi' \leq \xi .$$

Let us evaluate now $f(\xi) + g(\eta)$. Suppose first that $X(\eta) = Y^{-1}(\eta)$ is one valued. Then

(40) $$\begin{cases} f(\xi) + g(\eta) = g(\eta) + f(X(\eta)) - \int_\xi^{X(\eta)} 1_C\, f'(\xi')d\xi' - \\ \quad - \sum_i [f(\min(\xi_i',X(\eta))) - f(\max(\xi_i,\xi))] \end{cases}$$

where the summation is extended to the indices such that $[\xi_i,\xi_i']$ intersects $[\xi,X(\eta)]$. We have:

$$g(\eta) + f(X(\eta)) \geq 0 .$$

$$\int_\xi^{X(\eta)} 1_C\, f'(\xi')d\xi' \leq 0 \quad \text{by (39).}$$

As $X(\cdot)$ is one valued, it is not contained in the interior of an interval $[\xi_i,\xi_i']$. Thus

$$\min[\xi_i',X(\eta)) = \xi_i' \quad \text{if} \quad [\xi_i,\xi_i'] \cap [\xi,X(\cdot)] \neq \emptyset$$ and, if $\xi \notin [\xi_i,\xi_i']$, the corresponding term in the sum vanishes. For ξ in $[\xi_i,\xi_i']$, the term in the sum is

$$f(\xi_i') - f(\xi) ,$$

which is not positive, by (3.7). Therefore, the expression (40) is non-negative for $\xi \leq X(\eta)$. If we suppose that $X(\eta) = [\xi_j,\xi_j']$, we have to study the expression

$$g(\eta) + f(\xi_j') - \int_\xi^{\xi_j'} 1_C f'(\xi')d\xi' - \sum_i [f(\min(\xi_i',\xi_j')) - f(\max(\xi_i,\xi))]$$

and the result still holds, i.e.:

(41) $$w(x,t) \geq 0 \quad \text{for} \quad t \leq \tau(x).$$

Thanks to (41), we may redefine k as

$$k(x,t) = \inf\{w(x',t')/t' \leq t - |x-x'|\},$$

or still, in characteristic coordinates,

(42) $$\tilde{k}(\xi,\eta) = \inf\{f(\xi') + g(\eta')/\xi' \leq \xi \ \& \ \eta' \leq \eta\}.$$

Then, it is immediate that

(43) $$\tilde{k}(\xi,\eta) = \inf\{f(\xi')/\xi' \leq \xi\} + \inf\{f(\eta')/\eta' \leq \eta\},$$

which proves the claim of Lemma 4. ∎

We shall now prove that u, defined by (28) satisfies the transmission condition across the line of influence.

Lemma 5. If u is defined by (28), then almost everywhere on $\{x/|\tau'(x)| < 1\}$,

(44) $$\frac{\partial u}{\partial t}(x,\tau(x)+0) = -\frac{\partial u}{\partial t}(x,\tau(x)-0).$$

Proof. Let $A = \{x/|\tau'(x)| < 1\}$. Then, almost everywhere on A, by Corollary A.2 of [1],

(45) $$w_x(x,\tau(x)) \quad \text{and} \quad w_t(x,\tau(x)) \quad \text{exist.}$$

Let x be a point satisfying (45), and let us denote

$$w_x(x,\tau(x)) = a, \quad w_t(x,\tau(x)) = b, \quad \tau'(x) = m.$$

Then,

$$a + mb = 0, \quad b \leq 0$$

and

$$w(x',t') = a(x'-x) + b(t' - \tau(x)) + \varepsilon(x'-x, t' - \tau(x))$$

where

$$\lim_{|r|+|s|\to 0} \frac{\varepsilon(r,x)}{|r|+|s|} = 0.$$

We have

$$\inf\{a(x'-x) + b(t' - \tau(x))/(x',t') \in T^-_{x,t}\} = b(t - \tau(x))$$

and therefore,

(46) $$\begin{cases} b(t - \tau(x)) - \sup\{|\varepsilon(x'-x,t'-t)|/\tau(x') \leq t' \leq t - |x-x'|\} \leq \\ \quad \leq k(x,t) \leq b(t - \tau(x)) + |\varepsilon(0,t - \tau(x))|. \end{cases}$$

As $|\tau'(x)| < 1$, we have

$$\lim_{t \downarrow \tau(x)} [\sup\{|\varepsilon(x'-x,t'-t)/\tau(x') \leq t' \leq t - |x-x'|/t - \tau(x)] = 0$$

and we deduce from (46) that

$$\lim_{t \downarrow \tau(x)} \frac{u(x,t) - u(x,\tau(x))}{t - \tau(x)} = - w_t(x,\tau(x))$$

under the assumption (45). ∎

<u>Conclusion of the proof of Theorem 1.</u>

Lemmas 3, 4 and 5 imply that the function u defined by (28) solves the linear problem (24), up to the condition $u \in V$. Therefore it remains to check this last condition. If we take into account the formula (43), let us show that k is in V .

We know that f is in $H^1_{loc}(\mathbb{R})$; let

$$\hat{f}(\xi) = \inf\{f(\xi')/\xi' \leq \xi\}.$$

Then, we can compute the derivative of $\hat{f}(\xi)$ almost everywhere:

$$(47) \qquad \begin{cases} \hat{f}'(\xi) = 0 & \text{if} \quad f(\xi) > \hat{f}(\xi) \quad \text{or if} \quad f'(\xi) \geq 0 \\ f'(\xi) = f'(\xi) & \text{if} \quad f(\xi) = \hat{f}(\xi) \quad \text{and if} \quad f'(\xi) < 0 \; . \end{cases}$$

We deduce from (47) that $\hat{f}$ is in $H^1_{loc}(\mathbb{R})$. Similarly, $\hat{g}$ is in $H^1_{loc}(\mathbb{R})$. The function k which can be written as

$$k(x,t) = \hat{f}\left(\frac{x+t}{\sqrt{2}}\right) + g\left(\frac{-x+t}{\sqrt{2}}\right)$$

will therefore be in V , i.e.

$$\int_{-a}^{a} (|k_x(x,t)|^2 + |k_t(x,t)|^2)dx \leq C(a,b) \qquad \forall\, a,b,\ \forall\, t \leq 0$$

and thus u is in V . ∎

II.2. Continuous dependence on the data

Corollary 6. The map $(u_0,u_1) \to u$ which to an element of $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$ satisfying the compatibility condition (22) associates the solution of P_∞ is continuous from $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$ equipped with the strong topology to

$$W^{1,p}_{loc}([0,+\infty);\ L^2_{loc}(\mathbb{R})) \cap L^p_{loc}([0,+\infty);\ H^1_{loc}(\mathbb{R}))$$

equipped with the strong topology, for all finite p.

Proof. We have at once the continuity from $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$ to $C^0(\mathbb{R} \times \mathbb{R}^+)$.

The topology on $W^{1,p}_{loc}([0,+\infty);\ L^2_{loc}(\mathbb{R})) \cap L^p_{loc}([0,+\infty);H^1_{loc}(\mathbb{R}))$ is defined by the semi-norms for $A,B > 0$

$$q_{ABp}(u) = |u(0,0)| + \left(\int_0^B \left(\int_{-A}^A (u_x^2 + u_t^2)(x,t)dx\right)^{p/2}\right)^{1/p} .$$

The topology on $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$ is defined by the semi-norms for $A > 0$

$$p_A(u_0,u_1) = |u_0(0)| + \left(\int_{-A}^A \left(\left|\frac{du_0}{dx}\right|^2 + |u_1|^2\right)dx\right)^{1/2} .$$

It has been proved in [1], paragraph IV.2 that for solutions of P_∞ with zero obstacle

$$(48) \qquad \begin{cases} \left|\dfrac{\partial u}{\partial \xi}(x,t)\right| = \left|\dfrac{\partial u}{\partial \xi}(x+t,0)\right| \\ \left|\dfrac{\partial u}{\partial \eta}(x,t)\right| = \left|\dfrac{\partial u}{\partial \eta}(x-t,0)\right| \end{cases}$$

Therefore

$$(49) \qquad \begin{cases} \displaystyle\int_{-A}^A (|u_x|^2 + |u_t|^2)(x,t)dx = \int_{-A}^A (|u_\xi|^2 + |u_\eta|^2)(x,t)dx \le \\ \displaystyle\le \int_{-A-t}^{A+t} \left(\left|\frac{du_0}{dx}\right|^2 + |u_1|^2\right)dx \qquad \forall \quad A,t > 0 . \end{cases}$$

Let $q_{AB\infty}$ be the semi-norm

$$q_{AB\infty}(v) = v(0,0) + \operatorname*{ess\,sup}_{t\in[0,B]} \left(\int_{-A}^A (|v_x|^2 + |v_t|^2)(x,t)dx\right)^{1/2} .$$

Then, (49) implies

$$(50) \qquad q_{AB\infty}(u) \le p_{A+B}(u_0 + u_1) .$$

If (u_0^n,u_1^n) is a sequence of initial data satisfying the compatibility condition (22) and converging to (u_0,u_1) in $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$, then, as a consequence of (50),

$$(51) \qquad u^n \to u \quad \text{in} \quad H^1_{loc}(\mathbb{R} \times \mathbb{R}^+) \quad \text{weakly}$$

and moreover, (48) implies that

$$(52) \qquad \int_{-A}^{A} \left|\frac{\partial u^n}{\partial x}(x,t)\right|^2 dx \to \int_{-A}^{A} \left|\frac{\partial u}{\partial x}(x,t)\right|^2 dx \qquad \forall\ t,\ A > 0$$

$$(53) \qquad \int_{-A}^{A} \left|\frac{\partial u^n}{\partial t}(x,t)\right|^2 dx \to \int_{-A}^{A} \left|\frac{\partial u}{\partial t}(x,t)\right|^2 dx \qquad \forall\ t,\ A > 0 .$$

Gathering (51), (52) and (53), we obtain

$$(54) \qquad u^n \to u \quad \text{in} \quad H^1_{loc}(\mathbb{R} \times \mathbb{R}^+) \quad \text{strongly.}$$

Thanks to Fubini's theorem, one has from (54)

$$(55) \qquad \begin{cases} (u^n_x(\cdot,t),\ u^n_t(\cdot,t)) \to (u_x(\cdot,t), u_t(\cdot,t)) \\ \text{in } (L^2_{loc}(\mathbb{R}))^2 \text{ strongly, for almost all } t \ge 0 . \end{cases}$$

The relation (55) together with the estimate

$$q_{AB}(u^n) \le \sup_n q_{A+B}(u^n_0 + u^n_1) < +\infty$$

imply that u^n converges to u in the space $W^{1,p}_{loc}([0,+\infty);L^2_{loc}(\mathbb{R})) \cap L^p_{loc}([0,+\infty);H^1_{loc}(\mathbb{R}))$. ∎

Remark 7. The mapping $(u_0,u_1) \to u$ is not continuous to $W^{1,\infty}_{loc}([0,+\infty);L^2_{loc}(\mathbb{R})) \cap L^\infty_{loc}([0,+\infty);H^1_{loc}(\mathbb{R}))$ which is the space V defined in (17).

Take for instance the sequence of initial data

$$u^n_0 = 1 , \quad u^n_1 = -\frac{n+1}{n} .$$

As these do not depend on x, the solution of P is

$$u^n(x,t) = \begin{cases} 1 - \frac{n+1}{n}t & \text{if} \quad t \le \frac{n}{n+1}, \\ \frac{n+1}{n}t - 1 & \text{if} \quad t \ge \frac{n}{n+1}, \end{cases}$$

with the limit

$$u(x,t) = \begin{cases} 1 - t & \text{if} \quad t \le 1 \\ t - 1 & \text{if} \quad t \ge 1 . \end{cases}$$

Then we may calculate $q_{AB}(u^n - u)$:

$$q_{AB}(u^n - u) = 0 \qquad \text{if} \qquad B \le \frac{n}{n+1}$$

$$q_{AB}(u^n - u) = 2\sqrt{2A} \qquad \text{if} \qquad B > \frac{n}{n+1} .$$

Thus if $B \ge 1$, $q_{AB}(u^n - u)$ does not tend to zero as n tends to infinity. ∎

III.3. Application of the explicit formula to the finite string with fixed ends.

The explicit formula (29) will allow us to give a simple construction of the solution of P_f where P_f is the problem of the vibrating string with fixed ends, and obstacle $\psi = -K < 0$. The only modification with respect to P, is that we shall require u to be in the space $L^\infty(0,T;H^1_0(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L))$ for all $T > 0$.

In fact u will be in the space

$$L^\infty((0,\infty);\ H^1_0(0,L)) \cap W^{1,\infty}((0,\infty);\ L^2(0,L))$$

because we can integrate (10) on any rectangle $[0,L] \times [0,T]$, and we get the energy equality for arbitrary times T:

$$\int_0^L (|u_x(x,T)|^2 + |u_t(x,T)|^2)\,dx = \int_0^L \left(\left|\frac{du_0}{dx}\right|^2 + |u_1|^2 \right) dx \quad (56)$$

Let us define

$$e = \left\{ \int_0^L \left(\left|\frac{du_0}{dx}\right|^2 + |u_1|^2 \right) dx \right\}^{1/2} .$$

Then

$$|u(x,t) - u(0,t)| = \left|\int_0^x u_x(x',t)\,dx'\right| \leq e\sqrt{x}$$

and similarly

$$|u(x,t) - u(L,t)| \leq e\sqrt{L-x} .$$

Let $\delta = \dfrac{K^2}{e^2}$. Then

$$\forall t \in [0,\infty),\ \forall x \in [0,\delta) \cup (L-\delta,L],\ u(x,t) > -K \quad (57)$$

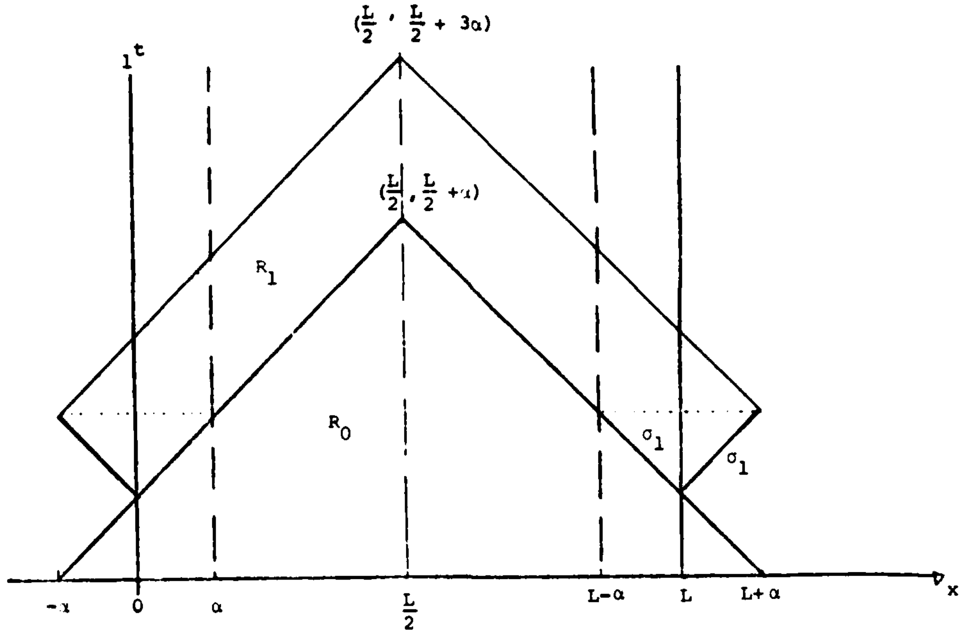
and ∂u cannot be supported in the strips $([0,\delta) \cup (L-\delta,L]) \times [0,\infty)$. Let us extend the initial conditions u_0, u_1 to the interval $[-\delta, L+\delta]$ by:

$$u_i(-x) = -u_i(x) \quad \text{if} \quad x \in [-\delta,0], \quad i = 0,1.$$

$$u_i(L+x) = -u_i(L-x) \quad \text{if} \quad x \in [0,\delta], \quad i = 0,1.$$

Then the corresponding free solution, w is defined on the cone $T^-_{\frac{L}{2},\frac{L}{2}+\delta}$, with the property that

$$w(x,t) = 0 \qquad 0 \leq t \leq \delta.$$



Let u be defined on $T^-_{\frac{L}{2}, \frac{L}{2}+\alpha}$ by (29); then for $x = 0$, $t \leq \alpha$

(58) $$u(0,t) = w(0,t) + 2 \sup_{T^-_{0,t}} \{[w(x',t') + K]^-\}.$$

But, $w(0,t) = 0$, and $T^-_{0,t}$ is included in the strip $[-\alpha,\alpha] \times [0,\infty)$, so that $w \geq -K$ on this strip, and thus $u(0,t) = 0$ on $[0,\alpha]$. Analogously, $u(L,t) = 0$ on $[0,\alpha]$.

Therefore, (5.8) defined the solution of P_f on $T^-_{\frac{L}{2}, \frac{L}{2}+\alpha} \cap ([0,L] \times [0,\infty))$.

Let us define by induction the solution of P_f on the region R_n given by:

(59) $$\begin{cases} R_n = \{(x,t) \in [0,L] \times [0,\infty) / \frac{L}{2} + (2n-1)\alpha - |x - \frac{L}{2}| \leq t \leq \\ \qquad \leq \frac{L}{2} + (2n+1)\alpha - |x - \frac{L}{2}|)\}. \end{cases}$$

We shall denote by σ_n the function

$$\begin{cases} \sigma_n(x) = \frac{L}{2} + (2n-1)\alpha - |x - \frac{L}{2}| & \text{if} \quad x \in [0,L] \\ \sigma_n(x) = \sigma_n(-x) \qquad \sigma_n(L + x) = \sigma_n(L - x) & \text{if} \quad x \in [0,\alpha]. \end{cases}$$

Suppose we know $u(x,\sigma_n(x))$ for $x \in [0,L]$. Let

$$
(60) \qquad \begin{cases} w_n(x,\sigma_n(x)) = u(x,\sigma_n(x)) & \text{if} \quad x \in [0,L] \\ w_n(x,\sigma_n(x)) = -u(x,\sigma_n(x)) & \text{if} \quad x \in [0,\ ] \\ w_n(x,\sigma_n(x)) = -u(2L - x,\ \sigma_n(2L - x) & \text{if} \quad x \in [L;\ L + \alpha] \\ \Box\, w_n = 0 \end{cases}
$$

The function w_n is defined in the region

$$
\sigma_n(x) \le t \le (2n + 1)\alpha + \frac{L}{2} - \left| x - \frac{L}{2} \right|, \quad -\alpha \le x \le L + \alpha.
$$

Let us notice that the symmetry of the initial conditions in (60) implies

$$
(61) \qquad w_n(0,t) = w_n(L,t) = 0 \quad \text{for} \quad (2n-1)\alpha \le t \le (2n+1)\alpha.
$$

Moreover, as $u\ (x,\sigma_n(x)) \ge -K$, $\forall\, x \in [0,L]$, and as u satisfies the energy condition (10), we shall have

$$
(62) \qquad \begin{cases} w_n(x,t) \ge -K \quad \text{for} \quad \sigma_n(x) \le t \le (2n+1)\alpha + \ \ - |x| \quad \text{or for} \\ \sigma_n(x) \le t \le (2n+1)\alpha + \ \ - |L - x|. \end{cases}
$$

Let

$$
(63) \qquad u(x,t) = w_n(x,t) + 2\sup\{(w_n(x',t') + K)^- / \sigma_n(x') \le t' \le t - |x - x'|\}.
$$

Thanks to (61) and (62), u satisfies the boundary conditions. Therefore it solves the problem of the string with an obstacle on R_n, and the induction can be pursued. ∎

III. A NUMERICAL SCHEME

III.1. A numerical scheme in a backward cone for the zero obstacle.

Let there be given initial data u_0 and u_1 on the interval $[-T,T]$. We seek an approximation to the problem $P_{0,T}$ on the backward cone $T^-_{0,T}$.

Let $h = \frac{T}{n}$ be a step, and let us define discretized initial data u^n_0 and u^n_1 by the following formula

$$
(64)\qquad \begin{cases} u^h_0(x) = \frac{1}{h}\,[u_0((p+1)h) - u_0(ph)]\,(x-ph) + u_0(ph) & \text{if } x \in [ph,(p+1)h] \\[2ex] u^h_1(x) = \frac{1}{h}\displaystyle\int_{ph}^{(p+1)h} u_1(x')\,dx' & \text{if } x \in [ph,(p+1)h]. \end{cases}
$$

The corresponding free solution w^h is given by

$$
w^h(x,t) = \frac{1}{2}\,[u^h_0(x+t) + u^h_0(x-t) + \int_{x-t}^{x+t} u^h_1(x')\,dx'] .
$$

Let us define

$$
(65)\qquad \tilde{w}^h_{i,j} = w^h\left(\frac{i-j}{2}\,h, \frac{i+j}{2}\,h\right) \quad \text{for } 0 \le \frac{i+j}{2} \le n - \left|\frac{i-j}{2}\right| .
$$

Then $\tilde{w}^h_{i,j}$ satisfies the finite difference relation

$$
(66)\qquad \tilde{w}^h_{i,j} = \tilde{w}^h_{i,j-1} + \tilde{w}^h_{i-1,j} - \tilde{w}^h_{i-1,j-1} .
$$

Let us define a function $\tilde{u}^h_{i,j}$ on our mesh by

$$
(67)\qquad \tilde{u}^h_{i,j} = \tilde{w}^h_{i,j} + 2\max\{(\tilde{w}^h_{i',j'})^- \,/\, i' \le i,\ j' \le j,\ i'+j' \ge 0\}.
$$

We could define $\tilde{u}^h_{i,j}$ alternatively by

$$
(68)\qquad \tilde{u}^h_{i,j} = \tilde{w}^h_{i,j} - 2\,\tilde{K}^h_{i,j}
$$

$$
(69)\qquad \begin{cases} \tilde{K}^h_{i,j} = \min(\tilde{K}^h_{i-1,j}\,,\ \tilde{K}^h_{i,j-1}\,,\ -(\tilde{w}^h_{i,j})^-) \\[1ex] \tilde{K}^h_{i,-i} = 0 \quad \text{if } -n \le i \le n . \end{cases}
$$

Notice that $\tilde{K}^h$ is not the discretization of k, but the discretization of $\tilde{k}.1_{\tilde{I}}$, where $\tilde{I}$ is the set I in characteristic coordinates.

<u>Theorem 8.</u> Let u be the solution of $P_{0,T}$ with zero obstacle, and let $\tilde{u}^h_{i,j}$ be defined by (67). Then:

$$
(70)\qquad \max_{i,j}\left|\tilde{u}^h_{i,j} - u\left(\frac{i-j}{2}\,h, \frac{i+j}{2}\,h\right)\right| \le C\sqrt{h}
$$

where C depends only on the initial conditions.

Moreover, we have the following bounds on the (approximate) characteristic derivatives:

(71) $$|\tilde{u}^h_{i,j} - \tilde{u}^h_{i-1,j}| \leq \frac{1}{2} \left| \int_{(i-1)h}^{ih} (u_{0x} + u_1)(x')dx' \right|$$

(72) $$|u^h_{i,j} - u^h_{i,j-1}| \leq \frac{1}{2} \left| \int_{-jh}^{(-j+1)h} (u_{0x} - u_1)(x')d' \right|$$

Proof. Let us first evaluate $w^h_{i,j} - w(x',t')$ when (s',t') is in the characteristic square centered on $(\frac{i-j}{2}h, \frac{i+j}{2}h)$, with sides of length $h\sqrt{2}$, i.e.

$$\frac{2i-1}{2}h \leq x' + t' \leq \frac{2i+1}{2}h \quad \text{and} \quad \frac{-2j-1}{2}h \leq x' - t' \leq \frac{-2j+1}{2}h .$$

$$\tilde{w}(x',t') - \tilde{w}^h_{i,j} = w(x',t') - w^h(\frac{i-j}{2}h, \frac{i+j}{2}h) =$$

$$= \frac{1}{2} [u_0(x' + t') - u^h_0(ih) + u_0(x' - t') - u^h_0(-jh) +$$

$$+ \int_{x'-t'}^{x'+t'} u_1(y)dy - \int_{-jh}^{ih} u^h_1(y)dy].$$

i.e.

$$|w(x',t') - \tilde{w}^h_{i,j}| \leq \frac{1}{2}\sqrt{\frac{h}{2}} [|\int_{ih}^{x'+t'} (u_{0x} + u_1)^2 dy|^{1/2} + |\int_{-jh}^{x'-t'} (u_{0x} - u_1)^2 dy|^{1/2}] \leq$$

$$\leq \frac{1}{2}\sqrt{\frac{h}{2}} [2 \int_{-T}^{T} [(u_{0x} + u_1)^2 + (u_{0x} - u_1)^2]dy]^{1/2} =$$

$$= \sqrt{\frac{h}{2}} (\int_{-T}^{T} (u^2_{0x} + u^2_1)dx)^{1/2} .$$

We may then deduce from

(73) $$|w(x',t') - \tilde{w}^h_{i,j}| \leq \sqrt{\frac{h}{2}} (\int_{-T}^{T} (u^2_{0x} + u^2_1)dx)^{1/2}$$

that

(74) $$\begin{cases} |\sup\{(\tilde{w}^h_{i,j})^- / i' \leq i,\ j' \leq j,\ i+j \geq 0\} - \sup\{[w(x',t')]^- / 0 \leq t' < t - |x-x'|\}| \leq \\ \leq \sqrt{\frac{h}{2}} (\int_{-T}^{T} (u^2_{0x} + u^2_1)dx)^{1/2} . \end{cases}$$

Let us notice that $w^h_{i,j} = w(\frac{i-j}{2}h, \frac{i+j}{2}h)$, because the approximation (64) is very particular.

This, in turn, gives

(75) $$\max_{i,j} |\tilde{u}^h_{i,j} - u(\frac{i-j}{2}h, \frac{i+j}{2}h)| \leq \sqrt{2h} (\int_{-T}^{T} (u^2_{0x} + u^2_1)dx)^{1/2}$$

This completes the proof of (70).

We now turn to prove (71) and (72). Let us notice first that if

$$\tilde{k}^h_{i,j} = \min\{\tilde{w}^h_{i',j'}/i' \le i,\ j' \le j\ ,\ i' + j' \ge 0\}.$$

We can write $k^h_{i,j}$ alternatively as

(76) $$\tilde{k}^h_{i,j} = \min\{\tilde{w}^h_{i',j'}/i' \le i,\ j' \le j\},$$

because we know from (41) that $\tilde{w}^h_{i,j} \ge 0$ for $i+j \le 0$, as long as we suppose that the domain of influence is not empty.

Relation (76) implies that

(77) $$\tilde{k}^h_{i,j} = \hat{f}^n(i) + \hat{g}^h(j)$$

where

(78) $$\tilde{w}^h_{i,j} = f^h(i) + g^h(j) \ ,$$

and

(79) $$\begin{cases} \hat{f}^h(i) = \min\{f^h(i')/i' \le i\}, \\ \hat{g}^h(j) = \min\{g^h(j')/j' \le j\}. \end{cases}$$

Thus, (68) can be written as

$$\tilde{u}^h_{i,j} = \tilde{w}^h_{i,j} + 2[\hat{f}^h(i) + \hat{g}^h(j)]^-$$

if T is not empty. If $\hat{f}^h(i) + \hat{g}^h(j) \ge 0$, then $\hat{f}^h(i-1) + \hat{g}^h(j) \ge 0$, and (71) is immediate. Suppose now that

(80) $$\hat{f}^h(i) + \hat{g}^h(j) < 0 \ .$$

We have two cases: in the first case,

(81) $$\hat{f}^h(i-1) + \hat{g}^h(j) \ge 0 \ .$$

Then, necessarily

(82) $$f^h(i) = \hat{f}^h(i) < \hat{f}^h(i-1) \le f^h(i-1)$$

and thus,

$$\tilde{u}^h_{i,j} - \tilde{u}^h_{i-1,j} = f^h(i) + g^h(j) - 2\hat{f}^h(i) - 2\hat{g}^h(j) -$$
$$- f^h(i-1) - g^h(j) = -[f^h(i) + \hat{g}^h(j) + \hat{f}^h(i-1) + \hat{g}^h(j)].$$

Thanks to (80) and (82), we get

(83) $$\begin{cases} |\tilde{u}^h_{i,j} - \tilde{u}^h_{i-1,j}| \le |f^h(i) + \hat{g}^h(j)| + |f^h(i-1) + \hat{g}^h(j)| \le \\ \qquad \le |f^h(i) - f^h(i-1)| \ . \end{cases}$$

In the second case,

$$\hat{f}^h(i-1) + \hat{g}^h(j) < 0 ,$$

if $\hat{f}^h(i-1) = \hat{f}^h(i)$, we have immediately

$$|\tilde{u}^h_{i,j} - \tilde{u}^h_{i-1,j}| \le |f^h(i) - f^h(i-1)|. \tag{84}$$

If $\hat{f}^h(i-1) > \hat{f}^h(i)$, then, we have (82), and

$$\tilde{u}^h_{i,j}-\tilde{u}^h_{i-1,j} = f^h(i) + g^h(j) - 2\hat{f}^h(i) - 2\hat{g}^h(j) - f^h(i-1) - g^h(j) + 2\hat{f}^h(i-1) + 2\hat{g}^h(j) =$$

$$= 2\hat{f}^h(i-1) - f^h(i) - f^h(i-1),$$

and, thanks to (82) we have

$$|\tilde{u}^h_{i,j} - \tilde{u}^h_{i-1,j}| \le |f^h(i) - f^h(i-1)|.$$

From (83), (84) and (85), we deduce

$$|\tilde{u}^h_{i,j} - \tilde{u}^h_{i-1,j}| \le |w^h_{i,j} - w^h_{i-1,j}| = \frac{1}{2} \left| \int_{(i-1)h}^{ih} (u_{0x} + u_1)(x')dx' \right|.$$

The proof of (72) is analogous. ∎

We can deduce from (71) and (72) an energy inequality: let i_0, j_0 be given such that $-n \le i_0, j_0 \le n$ and $i_0 + j_0 \ge 0$. Then we have:

$$\left\{ \begin{aligned} &\sum_{i=-j_0+1}^{i_0} \frac{1}{h}|\tilde{u}^h_{i,j_0} - \tilde{u}^h_{i-1,j_0}| + \sum_{j=-i_0+1}^{j_0} \frac{1}{h} |\tilde{u}^h_{i_0,j} - \tilde{u}^h_{i_0,j-1}|^2 \le \\ &\le \frac{1}{2} \int_{-j_0 h}^{i_0} [|u_{0x}|^2(x') + |u_1|^2(x')]dx' . \end{aligned} \right. \tag{86}$$

III.2 <u>A numerical scheme for the string with fixed ends and a constant obstacle.</u>

We shall use here the inductive construction of paragraph II.3, which we discretize.

Let u_0 and u_1 be given on $[0,L]$, and let

$$\alpha = K^2 / \left[\int_0^L \left(\left| \frac{du_0}{dx} \right|^2 + |u_1|^2 \right) dx \right], \tag{87}$$

where the obstacle is $\psi(x) = -K < 0$.

Let n be an even integer, and let the step be $h = \frac{L}{n}$; let n_0 be the largest integer such that $n_0 h \leq \alpha$.

We discretize the initial data as in (64) for $0 \leq p \leq n$, and we continue them by periodicity and imparity:

$$u_r^h(x) = -u_r^h(-x) \quad \text{for} \quad -n_0 h \leq x \leq 0;\ r = 0,1;$$

$$u_r^h(x) = -u_r^h(-x) \quad \text{for} \quad nh \leq x \leq (n+n_0)h;\ r = 0,1.$$

We define $w^{0,h}$ by

$$\begin{cases} w^{0,h}(x,0) = u_0^h(x) & -n_0 h \leq x \leq (n+n_0)h \\ \dfrac{\partial w^{0,h}}{\partial t}(x,0) = u_1^h(x) & -n_0 h \leq x \leq (n+n_0)h \\ \Box w^{0,h} = 0 & \text{in } T^-_{\frac{n}{2}h,\,(\frac{n}{2}+n_0)h} \end{cases} \tag{88}$$

and let

$$w_{i,j}^{0,h} = w^{0,h}\left(\frac{i-j}{2} h, \frac{i+j}{2} h \right). \tag{89}$$

Let

$$\tilde{u}_{i,j}^h = \tilde{w}_{i,j}^{0,h} + 2 \sup\{(\tilde{w}_{i,j}^{0,h} + K)^- / i' \leq i,\ j' \leq j,\ i'+j' \geq 0 \tag{90}$$

where $i \leq n+n_0$, $j \leq n_0$, $i+j \geq 0$.

Let us define a subset $R^{m,h}$ of $\mathbb{Z} \times \mathbb{Z}$ by

$$\begin{cases} R^{m,h} = [n + (2m-1)n_0, n + (2m+1)n_0] \times [-n + (2m-1)n_0, (2m-1)n_0] \\ \qquad [(2m-1)n_0, n + (2m+1)n_0] \times [(2m-1)n_0, (2m+1)n_0]. \end{cases} \tag{91}$$

The region $R^{m,h}$ is the discretized equivalent (in i,j coordinates) of the region R^m defined by (59).

We define $w^{m,h}$ on the lower boundary of $R^{m,h}$ by

$$(92)\quad \begin{cases} \tilde{w}^{m,h}_{i,j} = \tilde{u}^{n}_{i,j} & \text{for} \quad i = n + (2m-1)n_0,\ -n + (2m-1)n_0 \le j \le (2m-1)n_0 , \\ \tilde{w}^{m,h} = \tilde{u}^{h}_{i,j} & \text{for} \quad j = (2m-1)n_0,\ (2m-1)n_0 \le i \le n + (2m-1)n_0 \\ \tilde{w}^{m,h}_{i,j} = \tilde{w}^{m,h}_{j,i} & \text{for} \quad i = (2m-1)n_0,\ (2m-1)n_0 \le j \le (2m+1)n_0 , \\ \tilde{w}^{m,h}_{i,j} = \tilde{w}^{m,h}_{n+j,-n+i} & \text{for} \quad (2m\_1)n_0+n \le i \le (2m+1)n_0+n,\ j = -n+(2m-1)n \end{cases}$$

and in $R^{m,h}$, we have

$$(93)\qquad w^{m,h}_{i,j} = w^{m,h}_{i-1,j} + w^{m,h}_{i,j-1} - w^{m,h}_{i-1,j-1} \quad \text{for} \quad (i,j)\ \&\ (i-1,j-1) \quad \text{in} \quad R^{m,h}.$$

Then, we shall define $\tilde{u}^{h}_{i,j}$ on $R^{m,h} \cap \{(i,j)/0 \le \frac{i-j}{2} \le n\}$ by

$$(94)\qquad \tilde{u}^{h}_{i,j} = \tilde{w}^{m,h}_{i,j} + 2\sup\{(w^{m,h}_{i',j'} + K)^{-}/i' \le i,\ j' \le j \ \text{and}\ (i',j') \in R^{m,h}\}.$$

Of course (94) is the discretization of (63).

Theorem 9. Let u^h be defined by (93), and let u be the solution of P_f on $[0,L]$ with obstacle $-K$. Then

$$(95)\qquad \max_{i,j} |\tilde{u}^{h}_{i,j} - u(\tfrac{i-j}{2}h, \tfrac{i+j}{2}h)| \le C^{m+1}\sqrt{h}$$

for (i,j) in the region $R^{m,h}$ defined by (91), where C depends only on the initial conditions. Moreover we have the following bounds on the (approximate) characteristic derivatives:

$$(96)\quad \begin{cases} |\tilde{u}^{h}_{i,j} - \tilde{u}^{h}_{i-1,j}| \le \frac{1}{2} \left| \int_{(i-1)h}^{ih} (u_{0x} + u_1)(x')dx' \right| \\ |\tilde{u}^{h}_{i,j} - \tilde{u}^{h}_{i,j-1}| \le \frac{1}{2} \left| \int_{-jh}^{(-j+1)h} (u_{0x} - u_1)(x')dx' \right| \end{cases}$$

if u_0 and u_1 are extended to all $\mathbb{R}$ by periodicity and imparity.

Proof. We shall replace the number α defined in (87) by n_0h ; for this new value of α, we can perform the construction of the solution of P_f as in II.3, and we shall compare w^m and $\tilde{w}^{m,h}_{i,j}$ on the regions R^m and $R^{m,h}$.

Thanks to Theorem 8, the relation (95) is verified for $m = 0$ and $C \le (2\int_{-\alpha}^{L+\alpha} (u_{0x}^2 + u_1^2)dx)^{1/2}$ and the relation (96) is satisfied in R_0.

Suppose that for a certain constant C, (95) and (96) are satisfied in $R^{m-1,h}$.

Then we have

$$(97) \qquad |\tilde{w}_{i,j}^{m,h} - w^m(\frac{i-j}{2}h, \frac{i+j}{2}h)| \leq C^m \sqrt{h}$$

for i,j on the lower boundary of $R^{m,h}$ which is the upper boundary of $R^{m-1,h}$.

Then we have

$$(98) \qquad |\tilde{w}_{i,j}^{m,h} - w^m(\frac{i-j}{2}h, \frac{i+j}{2}h)| \leq 5\, C^m \sqrt{h} \quad \text{in} \quad R^{m,h},$$

because $\tilde{w}_{i,j}^{m,h}$, (respectively $w^m(\frac{i-j}{2}h, \frac{i+j}{2}h)$) is the sum of at most five terms $\tilde{w}_{i',j'}^{m,h}$ (respectively $w^m(\frac{i'-j'}{2}h, \frac{i'+j'}{2}h)$) with i',j' on the lower boundary of $R^{m,h}$ (respectively $(\frac{i'-j'}{2}h, \frac{i'+j'}{2}h)$ on the lower boundary of R^m).

If we evaluate now the difference $\tilde{w}_{i,j}^{m,h} - w^m(x',t')$ when (x',t') is in the characteristic square centered on $\frac{i-j}{2}h, \frac{i+j}{2}h$ with sides of length $h\sqrt{2}$, we have

$$(99) \qquad |\tilde{w}_{i,j}^{m,h} - w^m(x',t')| \leq 5C^m\sqrt{h} + |w^m(\frac{i-j}{2}h, \frac{i+j}{2}h) - w^m(x',t')|$$

but we have for P_f the equivalent of (48), i.e.

$$|\frac{\partial u}{\partial \xi}(x,t)| = |\frac{\partial u}{\partial \xi}(x+t,0)| = |\frac{1}{\sqrt{2}}(u_{0x} - u_1)(x-t,0)|,$$

$$|\frac{\partial u}{\partial \eta}(x,t)| = |\frac{\partial u}{\partial \eta}(x-t,0)| = |\frac{1}{\sqrt{2}}(u_{0x} - u_1)(x-t,0)|,$$

if y_0 and u_1 are extended to all of $\mathbb{R}$ by *imparity and periodicity*.

Therefore,

$$(100) \qquad \begin{cases} |\frac{\partial w^m}{\partial \xi}(x,t)| = \frac{1}{\sqrt{2}}\,|(u_{0x} + u_1)(x+t,0)|, \\ |\frac{\partial w^m}{\partial \eta}(x,t)| = \frac{1}{\sqrt{2}}\,|(u_{0x} - u_1)(x-t,0)|. \end{cases}$$

Relation (100) allows us to evaluate $w^m(\frac{i-j}{2}h, \frac{i+j}{2}h) - w^m(x',t')$,

$$(101) \qquad |w^m(\frac{i-j}{2}h, \frac{i+j}{2}h) - w^m(x',t,)| \leq \sqrt{2h}\,\left(\int_{-\alpha}^{L+\alpha}(u_{0x}^2 + u_1^2)dx\right)^{1/2}.$$

Let us denote by E the number

$$E = \int_{-\alpha}^{L+\alpha}(u_{0x}^2 + u_1^2)dx.$$

Gathering relations (97), (99) and (101), we obtain:

$$|\tilde{u}_{i,j}^{h} - u(\frac{i-j}{2}h, \frac{i+j}{2}h)| \leq (15C^m + 2\sqrt{2E})\,\sqrt{h}.$$

Therefore, if we choose $C = 15 + 2\sqrt{2}\,E$, we have

$$15C^m + 2\sqrt{2E} \leq C^{m+1}.$$

The proof of (96) is immediate. ∎

Remark 10. For (i,j) in $R^{m,h}$, we have

$$\frac{i+j}{2} \geq (2m-1)n_0$$

and thus

$$1 + m \leq \left(\left(\frac{i+j}{2}h\right)/2n_0 h\right) + \frac{3}{2} .$$

Therefore, if $\left(\frac{i-j}{2}h, \frac{i+j}{2}h\right)$ converges to (x,t) as h goes to zero, we have from (95):

$$\left|\tilde{u}^h_{i,j} - u\left(\frac{i-j}{2}h, \frac{i+j}{2}h\right)\right| \leq C_1^{\frac{t}{2\alpha} + \frac{3}{2}} \sqrt{h}$$

for all $C_1 > C$, and for all h small enough.

IV. REGULARITY IN SPACES OF FUNCTION OF LOCALLY BOUNDED VARIATION

This paragraph is dedicated to proving the following result of regularity for an arbitrary concave obstacle ψ.

Theorem 10. Let u_0 and u_1 be elements of $H^1_{loc}(\mathbb{R})$ and $L^2_{loc}(\mathbb{R})$ respectively, such that

$$\frac{du_0}{dx} \text{ and } u_1 \text{ are locally of bounded variation.} \tag{102}$$

Suppose that u_0 and u_1 satisfy the compatibility condition (22), and that the obstacle is concave.

Then for all η, the function

$$\xi \to \frac{\partial\tilde{u}}{\partial\xi}(\xi,\eta)$$

defined on $[-\eta,+\infty)$ is locally of bounded variation, and analogously, for all ξ the function

$$\eta \to \frac{\partial\tilde{u}}{\partial\eta}(\xi,\eta)$$

defined on $[-\xi,+\infty)$ is locally of bounded variation.

Proof. We retain the notations of II.1:

$$U = \{x/w(x,\psi(x)) > 0\} = \bigcup_i \,]a_i,b_i[$$

$$(35)\quad \begin{cases} \xi_i = \dfrac{a_i+\psi(a_i)}{\sqrt{2}} & \xi'_i = \dfrac{b_i+\psi(b_i)}{\sqrt{2}} \\[2ex] \eta_i = \dfrac{-a_i+\psi(a_i)}{\sqrt{2}} & \eta'_i = \dfrac{-b_i+\psi(b_i)}{\sqrt{2}} \end{cases}$$

$$(36)\quad \begin{cases} Y(\xi) = \eta_i & \text{if } \xi \in [\xi_i,\xi'_i) \\ Y(\xi) = [\eta'_i,\eta_i] & \text{if } \xi = \xi'_i \end{cases}$$

$$C = \left(\bigcup_i [\xi_i,\xi'_i]\right)^c .$$

We have the following representation of the solution:

$$(103)\quad \begin{aligned} \tilde{u}(\xi,\eta) &= f(\xi) + g(\eta) && \text{for } \eta \geq Y(\xi) \\ \tilde{u}(\xi,\eta) &= \hat{f}(\xi) + \hat{g}(\eta) && \text{for } \eta \leq Y(\xi) \end{aligned}$$

with the transmission conditions:

$$(104)\quad f(\xi) + g(Y(\xi)) = \hat{f}(\xi) + \hat{g}(Y(\xi)) = \psi((\xi - Y(\xi))/\sqrt{2})$$

$$f'(\xi) + g'(Y(\xi)) = -[\hat{f}'(\xi) + \hat{g}'(Y(\xi))] \quad \text{if } Y \text{ is one valued} \tag{105}$$

$$\text{and} \quad 0 > Y'(\xi) > -\infty .$$

If we differentiate (104) with respect to ξ on C, we get

$$f'(\xi) + Y'(\xi)g'(Y(\xi)) = \hat{f}'(\xi) + Y'(\xi)\hat{g}'(Y(\xi)) = \frac{1}{\sqrt{2}}\varphi'((\xi - Y(\xi))/\sqrt{2})(1-Y'(\xi)) \tag{106}$$

(notice that Y is decreasing on C, and therefore almost everywhere differentiable).

For $Y'(\xi) = 0$, we deduce from (106) that

$$f'(\xi) = \hat{f}'(\xi) = \frac{1}{\sqrt{2}}\varphi'(\xi - Y(\xi)). \tag{107}$$

For $0 > Y'(\xi) > -\infty$, we deduce from (105) and (106) that $\hat{f}'(\xi) + f'(\xi) = \sqrt{2}\,\varphi'(\xi - Y(\xi)$ which contains (107). Therefore, we have

$$\hat{f}'(\xi) + f'(\xi) = \sqrt{2}\,\varphi'(\xi - Y(\xi) \quad \text{a.e. on } C \tag{108}$$

and, differentiating (104) on C^c:

$$\hat{f}'(\xi) = f'(\xi) \quad \text{a.e. on } C^c . \tag{109}$$

Let us denote by h the function

$$h(\xi) = \frac{\partial u}{\partial \xi}(\xi,\eta) \tag{110}$$

where η is fixed throughout the end of this proof, and let

$$\xi_0 = \sup\{\xi' \in X(\eta)\} .$$

Then, for $\xi \le \xi_0$,

$$h(\xi) = f(\xi)$$

and ξ_0 does not belong to any interval (ξ_i,ξ_i').

To evaluate the total variation of h on a given bounded interval $I = [a,b]$, we have to estimate:

$$\left\{\begin{aligned} &TV(h;I) = TV(h;I \cap (-\infty,\xi_0)) + TV(h;C \cap (\xi_0,+\infty)) + TV(h;C^c \cap (\xi_0,+\infty)) + |h(\xi_0+0) - \\ &h(\xi_0 - 0)| + \sum_{\{i/\xi_0 \le \xi_i \le b\}} [|h(\xi_i'+0) - h(\xi_i'-0)| + |h(\xi_i+0) - h(\xi_i-0)|] . \end{aligned}\right. \tag{111}$$

According to (108) and (109), we have:

$$\left\{\begin{aligned} &TV(h;I \cap (-\infty,\xi_0)) + TV(h;C \cap (\xi_0,+\infty)) + TV(h;C^c \cap (\xi_0,+\infty)) \le \\ &\le TV(f;I) + TV(\sqrt{2}\,\varphi'(\tfrac{\xi - Y(\xi)}{\sqrt{2}}) - f(\xi);I) . \end{aligned}\right. \tag{112}$$

By hypothesis, φ'' is positive, therefore φ' is increasing; as $\xi \mapsto \frac{\xi - Y(\xi)}{\sqrt{2}}$ is increasing, the right hand side of (112) is bounded.

The term $|h(\xi_0 + 0) - h(\xi_0 - 0)|$ is bounded, because (102) ensures that f, and therefore $\hat{f}$ is locally bounded.

The remaining term in (111) is the sum

(113) $$\sum_{\{i/\xi_0 \le \xi_i \le b\}} [|h(\xi_i' + 0) - h(\xi_i' - 0)| + |h(\xi_i + 0) - h(\xi_i - 0)|] ,$$

which could possibly contain an infinite number of terms. Using (108), and (109) we can write the terms of (113) as

(114) $$|f(\xi_i + 0) + f(\xi_i - 0) - \sqrt{2}\,\varphi'(\frac{\xi_i - \eta_i}{\sqrt{2}} - 0)| + |f(\xi_i' + 0) + f(\xi_i' - 0) - \sqrt{2}\varphi'(\frac{\xi_i' - \eta_i'}{\sqrt{2}} + 0) .$$

But we have the following inequalities, deduced from the definition of the line of influence and of the intervals $[\xi_i, \xi_i']$:

(115) $$\begin{cases} f(\xi_i - 0) - \frac{1}{\sqrt{2}}\varphi'(\frac{\xi_i - \eta_i}{\sqrt{2}} - 0) = a_i^- \le 0 \\ f(\xi_i + 0) - \frac{1}{\sqrt{2}}\varphi'(\frac{\xi_i - \eta_i}{\sqrt{2}} + 0) = a_i^+ \ge 0 \\ f(\xi_i' - 0) - \frac{1}{\sqrt{2}}\varphi'(\frac{\xi_i' - \eta_i'}{\sqrt{2}} - 0) = b_i^- \le 0 \\ f(\xi_i' + 0) - \frac{1}{\sqrt{2}}\varphi'(\frac{\xi_i' - \eta_i'}{\sqrt{2}} + 0) = b_i^+ \le 0 . \end{cases}$$

We can estimate (114) by

(116) $$\begin{cases} |a_i^+ + a_i^-| + |b_i^+ + b_i^-| + \frac{1}{\sqrt{2}}[|\varphi'(\frac{\xi_i - \eta_i}{\sqrt{2}} + 0) - \varphi'(\frac{\xi_i - \eta_i}{\sqrt{2}} - 0)| + \\ + |\varphi'(\frac{\xi_i' - \eta_i'}{\sqrt{2}} + 0) - \varphi'(\frac{\xi_i' - \eta_i'}{\sqrt{2}} - 0)|] . \end{cases}$$

But

$$|a_i^+ + a_i^-| + |b_i^+ + b_i^-| \le |a_i^+ + a_i^-| + |b_i^+ - a_i^+| + |a_i^+ + a_i^-| + |a_i^- - b_i^-| ,$$

and using the sign conditions (115),

(117) $$\begin{cases} |a_i^+ + a_i^-| + |b_i^+ + b_i^-| \le 2|a_i^+ + a_i^-| + |b_i^+ - a_i^+| + |b_i^- + a_i^-| \le \\ \le 4\ \mathrm{TV}(f(\xi) - \frac{1}{\sqrt{2}}\varphi'(\frac{\xi - \gamma(\xi)}{\sqrt{2}});\ [\xi_i, \xi_i']) . \end{cases}$$

Carrying (117) and (116) into (113), we obtain:

(118) $$\begin{cases} \sum_{\{i/\xi_0 \le \xi_i \le b\}} [|h(\xi_i' + 0) - h(\xi_i' - 0)| + |h(\xi_i + 0) - h(\xi_i - 0)|] \le \\ \le 4\ \mathrm{TV}(f; [\xi_0, \xi_0']) + 5\ \mathrm{TV}(\frac{1}{\sqrt{2}}\varphi'(\frac{\xi - \gamma(\xi)}{\sqrt{2}});\ [\xi_0, \xi_0']) \end{cases}$$

Here, $\xi_0' = \sup\{\xi_i'/\xi_i \leq b\}$. The same argument holds for the other characteristic derivative. The proof of Theorem 10 is complete; notice that we have proved, in fact, that locally, $TV(\frac{\partial \bar{u}}{\partial \xi}(\cdot,\eta),I)$ is a bounded function of η, for all bounded I. ∎

Remark 11. It is not true that under hypothesis (102), $\frac{\partial u}{\partial \xi}(\cdot,t)$ or $\frac{\partial u}{\partial \xi}(\cdot,t)$ are of bounded variation for all t.

To see it, let us consider the following example. Let

$$(119)\quad \begin{cases} w(x,t) = A-t - a(x+t)^4 \sin \frac{1}{x+t} & \text{if} \quad |x+t| \leq b \\ \quad\quad = A-r & \text{if} \quad |x+t| \geq b. \end{cases}$$

We choose b such that $\sin \frac{1}{b} = 0$, and a such that the curve

$$(120)\qquad t = A - a(x+t)^4 \sin \frac{1}{x+t}$$

has always a slope lesser than 1, for $|x+t| \leq b$. For this purpose, we differentiate (120) with respect to x:

$$t' = 4a(1+t')(x+t)^3 \sin \frac{1}{x+t} - a(x+t)^2 \cos \frac{1}{x+t} \cdot (1+t'),$$

and so,

$$(121)\qquad |t'| \leq a \frac{4b^3+b^2}{1-a(4b^3+b^2)}.$$

Clearly $|t'|$ can be made smaller than 1 if a is sufficiently small.

Then, we choose A large enough to have

$$w(x,0) = A - ax^4 \sin \frac{1}{x} > 0 \quad \text{for} \quad |x| \leq b.$$

Obviously, $\frac{du_0}{dx} = w_x(x,0)$ and $u_1 = w_t(x,0)$ are locally of bounded variation.

Thanks to (121), the line of influence is given by (120). We shall now see that $\frac{\partial u}{\partial \eta}(\cdot,A)$ is not of bounded variation. The straight line $t = A$ crosses the line of influence infinitely many times, at the points

$$x = \frac{1}{n\pi} - A \quad \text{for} \quad \left|\frac{1}{n\pi}\right| < b, \quad n \in \mathbb{Z}$$

and we have

$$\begin{aligned} \frac{\partial u}{\partial \eta}(x,A) = -1 \quad & \text{if} \quad x \in \left(\frac{1}{(2k+2)\pi} - A, \frac{1}{(2k+1)\pi} - A\right), \quad k > 0 \\ & \text{or if} \quad x \in \left(\frac{1}{(2k+1)} - A, \frac{1}{(2k+2)} - A\right), \quad k < 0 \\ \frac{\partial u}{\partial \eta}(x,A) = +1 \quad & \text{if} \quad x \in \left(\frac{1}{(2k+1)\pi} - A, \frac{1}{2k\pi} - A\right), \quad k > 0 \\ & \text{or if} \quad x \in \left(\frac{1}{2k\pi} - A, \frac{1}{(2k+1)\pi} - A\right), \quad k < 0. \end{aligned}$$

This function is not of bounded variation on any interval containing zero. ∎

V. CONVERGENCE OF THE PENALTY METHOD.

V.1. Weak convergence.

This paragraph is dedicated to a general (and unfortunately coarse!) study of the penalized problem

$$(122)\qquad \begin{cases} \Box u_\lambda - \frac{1}{\lambda}(u_\lambda - \varphi)^- = 0 \\ u_\lambda(x,0) = u_0(x) \\ \frac{\partial u_\lambda}{\partial t}(x,0) = u_1(x) \end{cases}$$

where $r^- = \sup(-r,0)$, and φ is an arbitrary continuous function of x, and u_0, u_1 satisfy the compatibility condition (22).

Let us mention that (122) possesses always a unique solution; to see is, it is enough to write (122) in the form of an integral equation, and to use Picard iterations.

Proposition 12. We have the following estimates for the solution u_λ of (122):

$$(123)\qquad \begin{cases} \int_a^b [|\frac{\partial u_\lambda}{\partial t}(x,\sigma(x))|^2 + |\frac{\partial u_\lambda}{\partial x}(x,\sigma(x))|^2 + 2(\frac{\partial u_\lambda}{\partial t}\frac{\partial u_\lambda}{\partial x})(x,\sigma(x))\sigma'(x)]dx \leq \\ \leq \int_a^b (|\frac{du_0}{dx}|^2 + |u_1|^2)dx \end{cases}$$

for all Lipschitz continuous σ with Lipschitz constant 1 such that $\sigma > 0$ on (a,b), $\sigma(a) = \sigma(b) = 0$.

$$(124)\qquad \int_{T^-_{x,t}} \frac{1}{\lambda}(u_\lambda(x',t') - \varphi(x'))^- \,dx'dt' \leq C(x,t,u_0,u_1)$$

where C does not depend on λ.

Proof. (i) Estimate (123).

We have the identity

$$\begin{cases} (\Box u_\lambda - \frac{1}{\lambda}(u_\lambda - \varphi)^-)\frac{\partial u_\lambda}{\partial t} = \frac{\partial}{\partial x}(-\frac{\partial u_\lambda}{\partial t}\frac{\partial u_\lambda}{\partial x}) + \frac{1}{2}\frac{\partial}{\partial t}(|\frac{\partial u_\lambda}{\partial x}|^2 + |\frac{\partial u_\lambda}{\partial x}|^2 + \\ \qquad + \frac{1}{\lambda}((u_\lambda - \varphi)^-)^2) = 0 . \end{cases}$$

Integrating on the region $\{(x,t)/a \leq x \leq b$ and $0 \leq t \leq \sigma(x)\}$, we obtain the identity

$$\int_a^b [|\frac{\partial u_\lambda}{\partial t}(x,\sigma(x))|^2 + |\frac{\partial u_\lambda}{\partial x}(x,\sigma(x))|^2 + 2\sigma'(x)(\frac{\partial u_\lambda}{\partial t}\frac{\partial u_\lambda}{\partial x})(x,\sigma(x)) +$$

$$+ \frac{1}{\lambda}((u_\lambda(x,\sigma(x)) - \varphi(x))^-)^2]dx = \int_a^b (|\frac{du_0}{dx}|^2 + |u_1|^2)dx ,$$

noticing that $(u_\lambda(x,0) - \varphi(x))^- = 0$ for all x. From here, (123) is immediate.

(ii). Estimate (124).

We integrate $\Box u_\lambda = \frac{1}{\lambda}(u_\lambda - \varphi)^-$ on the backward cone $T^-_{x,t}$:

$$\int_{T^-_{x,t}} \Box u_\lambda dx'dt' = \int_{x-t}^{x+t} (\frac{\partial u_\lambda}{\partial t}(x',t-|x-x'|) - u_1(x',0))dx' -$$

$$- \int_0^t (\frac{\partial u_\lambda}{\partial x}(x+t-t',t') - \frac{\partial u_\lambda}{\partial x}(x-t+t'))dt' =$$

$$= \int_{x-t}^x [\frac{\partial u_\lambda}{\partial t}(x',t-|x-x'|) + \frac{\partial u_\lambda}{\partial x}(x',t-|x-x'|)]dx' +$$

$$+ \int_x^{x+t} [\frac{\partial u_\lambda}{\partial t}(x',t-|x-x'|) - \frac{\partial u_\lambda}{\partial x}(x',t-|x-x'|)]dx' - \int_{x-t}^{x+t} u_1(x')dx' .$$

Let $\sigma(x') = t - |x-x'|$. Then

$$\int_{T^-_{x,t}} \Box u_\lambda(x',t')dx'dt' \le$$

$$\le \int_{x-t}^{x+t} [\frac{\partial u_\lambda}{\partial t}(x',\sigma(x')) + \sigma'(x')\frac{\partial u_\lambda}{\partial x}(x',\sigma(x'))]dx' + \int_{x-t}^{x+t} |u_1(x')|dx' ,$$

and using Schwarz inequality and (123), we obtain

$$\int_{T^-_{x,t}} \frac{1}{\lambda}(u_\lambda - \varphi)^- dx'dt' \le$$

$$\le \int_{x-t}^{x+t} [\frac{\partial u_\lambda}{\partial t}(x',\sigma(x')) + \sigma'(x')\frac{\partial u_\lambda}{\partial x}(x',\sigma(x'))]^2 dx']^{1/2}\sqrt{2t} + (\int_{x-t}^{x+t} |u_1(x')|^2 dx')^{1/w}\sqrt{2t} \le$$

$$\le 2\sqrt{2t}\,(\int_{x-t}^{x+t} |(u_1(x')|^2 + |\frac{du_0}{dx}(x')|^2)dx')^{1/2}. \qquad \blacksquare$$

We need definitions of left and right traces of the characteristic derivatives of a function u.

The following results were proved in [1]: let u be in V (cf. def. (17)), such that $\Box u$ is a positive measure. Then the function

$$\eta \mapsto \frac{\partial \tilde{u}}{\partial \xi}(\xi,\eta)\Big|_{[a,b]}$$

is increasing from $[-a,\infty)$ to $L^2(a,b)$ for all a,b, and similarly

$$\xi \mapsto \frac{\partial \tilde{u}}{\partial \eta}(\xi,\eta)\Big|_{[c,d]}$$

is increasing from $[-c,\infty)$ to $L^2(c,d)$ for all c,d.

We define

$$(125)\qquad \begin{cases} \dfrac{\partial \tilde{u}^r}{\partial \xi}(\xi,\eta) = \lim\limits_{h\downarrow 0} \dfrac{\partial \tilde{u}}{\partial \xi}(\xi,\eta + h) \\ \dfrac{\partial \tilde{u}^\ell}{\partial \xi}(\xi,\eta) = \lim\limits_{h\downarrow 0} \dfrac{\partial \tilde{u}}{\partial \xi}(\xi,\eta - h) \\ \dfrac{\partial \tilde{u}^r}{\partial \eta}(\xi,\eta) = \lim\limits_{h\downarrow 0} \dfrac{\partial \tilde{u}}{\partial \eta}(\xi + h,\eta) \\ \dfrac{\partial \tilde{u}^\ell}{\partial \eta}(\xi,\eta) = \lim\limits_{h\downarrow 0} \dfrac{\partial \tilde{u}}{\partial \eta}(\xi - h,\eta). \end{cases}$$

The functions $\dfrac{\partial \tilde{u}^r}{\partial \xi}$ and $\dfrac{\partial \tilde{u}^\ell}{\partial \xi}$ are defined for all ξ not belonging to the null set N_ξ, and for all η larger than $-\xi$; analogously, the functions $\dfrac{\partial \tilde{u}^r}{\partial \eta}$ and $\dfrac{\partial \tilde{u}^\ell}{\partial \eta}$ are defined for all η not belonging to the null set N_η and for all ξ larger than $-\eta$.

Proposition V.2 and Corollary V.4 of [1] tell us that

$$\frac{\partial u^r}{\partial \xi}(\cdot,\sigma(\cdot)) \in L^2_{loc}(\mathbb{R};(1+\sigma')dx)$$

$$\frac{\partial u^r}{\partial \eta}(\cdot,\sigma(\cdot)) \in L^2_{loc}(\mathbb{R};(1-\sigma')dx)$$

$$\frac{\partial u^\ell}{\partial \xi}(\cdot,\sigma(\cdot)) \in L^2_{loc}(\{x|\sigma(x) > 0\}, (1+\sigma')dx)$$

$$\frac{\partial u^\ell}{\partial \eta}(\cdot,\sigma(\cdot)) \in L^2_{loc}(\{x|\sigma(x) > 0\}, (1-\sigma')dx).$$

Notice that the above traces are not continuous functions of u. We have the following example:

$$u_n(x,t) = 1 + \frac{1}{n} - t \qquad \text{if} \quad t \le 1 + \frac{1}{n}$$

$$\qquad = t - (1 + \frac{1}{n}) \qquad \text{if} \quad t \ge 1 + \frac{1}{n}.$$

Then $\dfrac{\partial u^r}{\partial \xi}(x,1) = \dfrac{1}{\sqrt{2}}$, $\forall\, x$, and $\dfrac{\partial u_n^r}{\partial \xi}(x,1) = -\dfrac{1}{\sqrt{2}}$ $\forall x, \forall n$.

We may now state the following result of weak convergence of the penalization:

<u>Theorem 13.</u> Given initial conditions $u_0 \in L^1_{loc}(\mathbb{R})$ and $u_1 \in L^2_{loc}(\mathbb{R})$ such that $u_0 \geq \varphi$ and $u_1 \geq 0$ a.e. on the set $\{x | u_0(x) = \varphi(x)\}$, there exists a function u such that

(126) $\qquad u \in V$

(127) $\qquad u \geq \varphi$

(128) $\qquad \Box u \geq 0$

(129) $\qquad \operatorname{supp} \Box u \subset \{(x,t)/u(x,t) = \varphi(x)\}$

(130)
$$\left\{\begin{array}{l} \int_a^b [|\frac{\partial u^r}{\partial \xi}(x,\sigma(x))|^2(1+\sigma'(x)) + |\frac{\partial u^r}{\partial \eta}(x,\sigma(x))|^2(1-\sigma'(x))]dx \leq \\ \qquad \leq \int_a^b (|u_1|^2 + \left|\frac{du_0}{\partial x}\right|^2)dx \\ \int_a^b [|\frac{\partial u^\ell}{\partial \xi}(x,\sigma(x))|^2(1+\sigma'(x)) + \frac{\partial u^\ell}{\partial \eta}(x,\sigma(x))|^2(1-\sigma'(x))]dx \leq \\ \qquad \leq \int_b^a (|u_1|^2 + |\frac{du_0}{dx}|^2)dx \\ \text{for all Lipschitz continuous function } \sigma \text{, with Lipschitz constant } 1, \\ \text{such that } \sigma(a) = \sigma(b) = 0, \ \sigma > 0 \text{ on } (a,b). \end{array}\right.$$

(131) $\qquad u(x,0) = u_0(x)$

(132)
$$\left\{\begin{array}{lll} \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{if} & u_0(x) > \varphi(x) \\ |\frac{\partial u}{\partial t}(x,0)| \leq u_1(x) & \text{if} & u_0(x) = \varphi(x) \end{array}\right.$$

<u>Proof</u>. From estimates (123) and (124), we can see that we can extract a subsequence u_μ such that

(133) $\qquad u_\mu \to u \quad$ weakly* in V.

The weak * topology on V is defined by the semi norms

$$|\int u f_1| + |\int u_x f_2| + |\int u_t f_3|$$

where f_1, f_2 and f_3 are in $L^1(\mathbb{R}^+; L^2(\mathbb{R}))$ with compact support in $\mathbb{R} \times [0,\infty)$.

We deduce from (133) that

(134) $\qquad u_\mu \to u$ in $C^0(\mathbb{R} \times \mathbb{R}^+)$ with the compact topology.

Possibly with a new extraction

(135) $\qquad \frac{1}{\mu}(u_\mu - \varphi)^- \to \nu$ vaguely in $M(\mathbb{R}\times\mathbb{R}^+)$ the set of measures on $\mathbb{R}\times\mathbb{R}^+$.

Therefore

(136) $\qquad \Box u = \nu \geq 0.$

Relation (123) gives a bound on $\frac{((u_\mu - \varphi)^-)^2}{\mu}$ in L^1_{loc} and thus

$$u \geq \varphi.$$

To check (129), let (x_0,t_0) be a point such that $u(x_0,t_0) > \varphi(x_0)$; thanks to (134) we can find a neighborhood U of (x_0,t_0) and a μ_0 such that $u_\mu(x,t) - \varphi(x) \geq \frac{1}{4}(u(x_0,t_0) - \varphi(x_0))$ $\forall \mu < \mu_0$, $\forall (x,t) \in U$.

Therefore

$$\Box u_\mu|_U = 0 \quad \text{for } \mu < \mu_0,$$

and in the limit

$$\Box u|_U = 0 .$$

This proves (129).

To prove (130), let ψ be given, and ε_0 be a positive number. Let us define for $|\varepsilon| < \varepsilon_0$

(137) $$\psi_\varepsilon(x) = \begin{cases} \psi(x) + \varepsilon & \text{if } x \in [a+\varepsilon_0, b-\varepsilon_0] \\ x - a + \varepsilon - \varepsilon_0 & \text{if } x \in [a+\varepsilon_0-\varepsilon, a+\varepsilon_0] \\ -x + b - \varepsilon_0 + \varepsilon & \text{if } x \in [b-\varepsilon_0, b+\varepsilon-\varepsilon_0]. \end{cases}$$

Then (123) implies

$$\int_{-\varepsilon'}^{\varepsilon''} d\varepsilon \int_{a+\varepsilon_0}^{b-\varepsilon_0} \left[\left(\frac{\partial u_\mu}{\partial \xi}(x,\psi_\varepsilon(x))\right)^2(1+\psi'(x)) + \left(\frac{\partial u_\mu}{\partial \eta}(x,\psi_\varepsilon(x))\right)^2(1-\psi'(x))\right]dx \leq$$

$$C(\varepsilon' + \varepsilon'') \int_{a-\varepsilon_0}^{b+\varepsilon_0} \left(\left(\frac{du_0}{dx}\right)^2 + u_1^2\right)dx .$$

But the left hand side term of (137) can be written as

$$\int_{a+\varepsilon_0}^{b-\varepsilon_0} dx \int_{\psi(x)-\varepsilon'}^{\psi(x)+\varepsilon''} \left(\frac{\partial u_\mu}{\partial \xi}(x,t)\right)^2(1+\psi'(x)) + \left(\frac{\partial u_\mu}{\partial \eta}\right)^2(x,t)(1-\psi'(x))\,dt$$

and we can take a weak limit in this double integral thanks to (133).

Thus we can rewrite (138) without the index μ :

$$(139)\quad \begin{cases} \displaystyle\int_{-\varepsilon'}^{\varepsilon''} d\varepsilon \int_{a+\varepsilon_0}^{b-\varepsilon_0} [|\frac{\partial u}{\partial \xi}(x,\sigma_\varepsilon(x))|^2(1+\sigma_\varepsilon'(x)) + |\frac{\partial u}{\partial \eta}(x,\sigma_\varepsilon(x))|^2(1-\sigma_\varepsilon'(x))]dx \le \\ \displaystyle\le (\varepsilon' + \varepsilon'') \int_{a-\varepsilon_0}^{b+\varepsilon_0} (|\frac{du_0}{dx}|^2 + |u_1|^2)dx . \end{cases}$$

Taking $\varepsilon' = 0$ in (139) and letting ε'' tend to zero, we obtain

$$\int_{a+\varepsilon_0}^{b-\varepsilon_0} [|\frac{\partial u^r}{\partial \xi}(x,\sigma(x))|^2(1+\sigma'(x)) + |\frac{\partial u^r}{\partial \eta}(x,\sigma(x))|^2(1-\sigma'(x))]dx \le$$

$$\le \int_{a-\varepsilon_0}^{b+\varepsilon_0} (|\frac{du_0}{dx}|^2 + |u_1|^2)dx .$$

Letting ε_0 go to zero, we obtain the first relation of (130). If we take $\varepsilon'' = 0$ and let ε' , then let ε_0 tend to zero, we obtain the second relation of (130).

The initial condition (131) is obviously satisfied. It remains to check (132). For this purpose, let us take in (137) $\sigma(x) = 0$ on $[a,b]$. Then, ultimately we get

$$\int_a^b [|\frac{\partial u^r}{\partial \xi}(x,0)|^2 + |\frac{\partial u^r}{\partial \eta}(x,0)|]dx \le \int_a^b (|\frac{du_0}{dx}|^2 + |u_1|^2)dx.$$

Using the identity

$$|\frac{\partial u^r}{\partial \xi}(x,0)|^2 + |\frac{\partial u^r}{\partial \eta}(x,0)|^2 = |\frac{du_0}{dx}|^2 + |\frac{\partial u}{\partial t}(x,0+0)|^2$$

which takes into account (131), we have

$$\int_b^a |\frac{\partial u}{\partial t}(x,0+0)|^2 dx \le \int_a^b |u_1|^2 dx .$$

As a and b are arbitrary, we have eventually

$$|\frac{\partial u}{\partial t}(x,)+))| \le |u_1(x)| \quad \text{a.e. on } \mathbb{R}.$$

When $u_0(x) > \varphi(x)$, we have the first part of (132), as locally, $\nu = \Box u = 0$. ■

We shall now study the relation between the strong convergence of $\frac{\partial u_\lambda}{\partial x}$ and $\frac{\partial u_\lambda}{\partial t}$, and the verification of the energy condition (11).

<u>Lemma 14.</u> Let u_λ be a sequence of solutions of (122), converging weakly * to a solution u of (126)-(132). Then, u satisfies the energy condition (11) if and only if $\frac{\partial u_\lambda}{\partial t}$ and $\frac{\partial u_\lambda}{\partial x}$ converge to $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ respectively, strongly in $L^2_{loc}(\mathbb{R} \times [0,\infty))$.

__Proof.__ Notice first that as $\frac{(u_\mu - \varphi)^-}{\mu} \cdot 1_K$ converges to $\nu \cdot 1_K$ in $M(\mathbb{R} \times \mathbb{R}^+)$ weakly, for all compact K, and as $(u_\mu - \varphi)^-$ converges to zero uniformly on compact sets, then

$$(140) \qquad \int_K [((u_\mu - \varphi)^-)^2/\mu]dx'dt' \to 0$$

for any compact set K.

Let $\sigma_h(x') = h - |x-x'|$. Then, we have the identity, for any function v:

$$(141) \qquad \begin{aligned} &\int_0^t dh \int_{x-h}^{x+h} [|\frac{\partial v}{\partial \xi}(x,\sigma_h(x))|^2(1+\sigma_h'(x)) + |\frac{\partial v}{\partial \eta}(x,\sigma_h(x))|^2(1-\sigma_h'(x))]dx = \\ &= \int_{x-t}^{x} dx' \int_0^{t+x-x'} 2|\frac{\partial v}{\partial \xi}(x',t')|^2 dt' + \int_x^{x+t} dx' \int_0^{t-x+x'} 2|\frac{\partial v}{\partial \eta}(x',t')|^2 dt' . \end{aligned}$$

If the limit of the sequence u_μ satisfies (11), then the value of (141) $v = u$ is

$$(142) \qquad \int_0^t dh \int_{x-h}^{x+h} (|\frac{du_0}{dx}|^2 + |u_1|^2)dx ;$$

The value of (141) for $v = u$ is

$$(143) \qquad \int_0^t dh \int_{x-h}^{x+h} (|\frac{du_0}{dx}|^2 + |u_1|^2)dx - \int_{0\le t'\le t-|x-x'|} \frac{1}{\mu}((u_\mu - \varphi)^-)^2 dx'dt .$$

And according to (140), the limit of (143) is (142). Therefore, as $\frac{\partial u_\mu}{\partial \xi}$ (resp. $\frac{\partial u_\mu}{\partial \eta}$) converges weakly to $\frac{\partial u}{\partial \xi}$ (resp $\frac{\partial u}{\partial \eta}$) in $L^2_{loc}([0,\infty) \times \mathbb{R}^+)$, and

$$\lim_{\mu \to 0} \int_{A_{x,t}} |\frac{\partial u_\mu}{\partial \xi}|^2 dx'dt' + \int_{B_{x,t}} |\frac{\partial u_\mu}{\partial \eta}|^2 dx'dt' = \int_{A_{x,t}} |\frac{\partial u}{\partial \xi}|^2 dx'dt' + \int_{B_{x,t}} |\frac{\partial u}{\partial \eta}|^2 dx'dt'$$

where $A_{x,t} = \{(x',t') \in T^-_{x,t} / x' \le 0\}$, $B_{x,t} = T^-_{x,t} \setminus A_{x,t}$, we can conclude that the convergence of $\frac{\partial u_\mu}{\partial \xi}$ and $\frac{\partial u_\mu}{\partial \eta}$ to $\frac{\partial u}{\partial \xi}$ and $\frac{\partial u}{\partial \eta}$ is strong.

Conversely, if $\frac{\partial u_\mu}{\partial \xi}$ (resp. $\frac{\partial u_\mu}{\partial \eta}$) converges strongly to $\frac{\partial u}{\partial \xi}$ (resp. $\frac{\partial u}{\partial \eta}$), then it is straightforward to pass to the limit in (11). ∎

V.2. Strong convergence when the obstacle is zero and the initial characteristic derivatives are of bounded variation.

The first step in this study is to notice that if w is an affine function, then the penalized solution converges to the solution of (P_∞) which conserves the energy.

Lemma 15. Let there be given initial conditions

$$(144)\quad \begin{cases} u(x,0) = a - bx \geq 0 \quad \text{on} \quad [x_0-t_0, x_0+t_0] \\ u_t(x,0) = -c < 0 \end{cases}$$

and suppose that the free solution $w(x,t) = a - bx - ct$ is such that

$$w(x_0,t_0) < 0 .$$

Then the solution u_λ of (122) with initial conditions (144) is given by

$$(145)\quad \begin{cases} u_\lambda(x,t) = a - bx - ct & \text{for} \quad bx + ct < a , \\ u_\lambda(x,t) = \sqrt{\lambda(c^2 - b^2)}\sin \dfrac{ct + bx - a}{\sqrt{\lambda(c^2 - b^2)}} & \text{for} \quad a \leq bx + ct \leq a + r\sqrt{\lambda(c^2 - b^2)} \\ u_\lambda(x,t) = bx - ct - a - r\sqrt{\lambda(c^2 - b^2} & \text{for} \quad bx + ct \geq a + r\sqrt{\lambda(c^2 - b^2)}. \end{cases}$$

Therefore u_λ converges strongly in $H^1(T^-_{x_0,t_0})$ to the solution of P_{x_0,t_0} .

Proof. Let us compute the solution of P_{x_0,t_0}:

$$E = \{(x,t) \in T^-_{x_0,t_0} \;/\; a - bx - ct < 0\}.$$

We see at once that the slope of the line $a = bx + ct$ is smaller than 1 , in absolute value. Therefore $I = E$, and

$$(146)\quad \begin{cases} u(x,t) = a - bx - ct & \text{if} \quad a - bx - ct \geq 0 \\ u(x,t) = bx + ct - a & \text{if} \quad a - bx - ct \leq 0 . \end{cases}$$

Let us look for the solution of (122) with initial conditions (144) under the form

$$u_\lambda(x,t) = f_\lambda(bx + ct) .$$

Then f_λ must satisfy the ordinary differential equation

$$(c^2 - b^2)f'' - \frac{1}{\lambda} f^- = 0$$

with the initial conditions

$$\begin{cases} f_\lambda(z) = 0 \\ f'_\lambda(a) = -1 . \end{cases}$$

This problem can be solved immediately and gives (145). Clearly the limit of the sequence u_ϵ is u, and Lemma 14 allows us to conclude. ∎

Remark 16. Suppose we replace the function r^- by a function [illegible] such that

$$[illegible](x) = 0 \quad \text{if} \quad x \geq [illegible]$$

$$[illegible](x) \; [illegible] \quad \text{if} \quad x \; [illegible]$$

[illegible] is continuous, strictly decreasing on $(-\infty, [illegible])$

$$[illegible](-\infty) = [illegible]$$

then the penalized problem

$$\begin{cases} \square \hat{u}_\epsilon - \frac{1}{\epsilon}[illegible](\hat{u}_\epsilon - \varphi) = 0 \\ \hat{u}_\epsilon(x,0) = u_0(x) \\ \frac{\partial \hat{u}_\epsilon}{\partial t}(x,0) = u_1(x) \end{cases}$$

can be studied as above; we get Theorem 14 with almost no change in the proof. Moreover a phase plane analysis shows easily that in the case of initial data (144) the limit of u_ϵ is the function (146). We chose the specific penalization (122) because of its simplicity. We need an integral solution of the linear Klein-Gordon equation with initial values given on a curve $t = \varphi(x)$. It is the object of the next lemma.

Lemma 17. Let w be a solution of the wave equation on the set $S = \{(x,t)/\varphi(x) \leq t \leq t_0 - |x-x_0|\}$ where φ is a Lipschitz continuous function with Lipschitz constant 1.

Then the unique solution on S of the problem

$$(147) \qquad \begin{cases} \square u + \frac{1}{\epsilon} u = 0 \\ u(x,\varphi(x)) = w(x,\varphi(x)) \\ \frac{\partial u}{\partial t}(x,\varphi(x)) = \frac{\partial w}{\partial t}(x,\varphi(x)) \quad \text{a.e. on} \quad \{x/|\varphi'(x)| < 1\} \end{cases}$$

is given by

$$(148) \qquad u(x,t) = w(x,t) - \frac{1}{2\epsilon}\iint_{S \cap T_{x,t}} J_0\left(\frac{\sqrt{(t-t')^2 - (x-x')^2}}{\sqrt{\epsilon}}\right) w(x',t')\,dx'dt' ,$$

where

$$(149) \qquad J_0(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{y}{2}\right)^{2n}$$

is the Bessel function J_0.

Proof. We verify that if w is a solution of the wave equation in the whole plane, and if,

$$\bar{w}(x,t) = \begin{cases} 0 & \text{if} \quad t < \sigma(x) \\ w(x,t) & \text{if} \quad t \geq \sigma(x) \end{cases}$$

then,

$$\langle \Box \bar{w}, \varphi \rangle = - \int w(x,\sigma(x))(\varphi_t(x,\sigma(x)) + \sigma'(x)\varphi_x(x,\sigma(x)))dx +$$
$$+ \int [w_t(x, \ (x)) + \sigma'(x)w_x(x,\sigma(x))]\varphi(x,\sigma(x))dx.$$

Solving (147) amounts to find a solution of

$$\begin{cases} (\Box u + \frac{1}{\lambda}u)\Big|_{\{(x,t)/t > \sigma(x)\}} = 0 \\ u(x,t)\Big|_{\{(x,t)/t \leq \sigma(x)\}} = 0 \\ u(x,\sigma(x)) = w(x,\sigma(x)) \\ \frac{\partial u}{\partial t}(x,\sigma(x)) = \frac{\partial w}{\partial t}(x,\sigma(x)) \quad \text{a.e. on} \quad \{x/|\sigma'(x)| < 1\} \end{cases}$$

which can be written as

(150) $$u = \bar{w} - \frac{1}{\lambda} \mathcal{E} * u$$

where $\mathcal{E}$ is the elementary solution of the wave equation defined by

$$\mathcal{E}(x,t) = \begin{cases} \frac{1}{2} & \text{if} \quad t \geq x \\ 0 & \text{elsewhere} \end{cases}$$

The convolution equation (150) has a unique solution given by

(151) $$u = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^n} \left(\mathop{*}_{k=1}^{n} \mathcal{E}\right) * \bar{w} .$$

By a simple inductive calculation in characteristic coordinates, we obtain:

(152) $$\left(\mathop{*}_{k=1}^{n} \tilde{\mathcal{E}}\right)(\xi,\eta) = \tilde{\mathcal{E}} \cdot \left(\frac{\xi\eta}{2}\right)^{n-1} \frac{1}{((n-1)!)^2} .$$

Therefore

(153) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda^n} \left(\mathop{*}_{k=1}^{n} \mathcal{E}\right)(x,t) = - \frac{1}{2\lambda} \mathcal{E} \cdot J_0\left(\frac{\sqrt{t^2 - x^2}}{\lambda}\right)$$

Together with (153), formula (151) gives (149). ∎

We can now state the theorem for convergence for penalized solutions:

Theorem 18. Let u_0 and u_1 be such that $\frac{du_0}{dx}$ and u_1 are locally of bounded variation

(154) and suppose they satisfy the compability condition (22). Then the solution u_λ of (122 converges to the solution of P_α when λ goes to zero.

Proof. Let us first notice that on I^c, the complement of the domain of influence, we have, if u is the solution of P :

$$\left\{\begin{array}{l} \Box u = 0 \\ u \geq 0 \end{array}\right\} \quad \text{on } I^c .$$

Therefore

$$u_\lambda = u \quad \text{on } \overline{I^c}$$

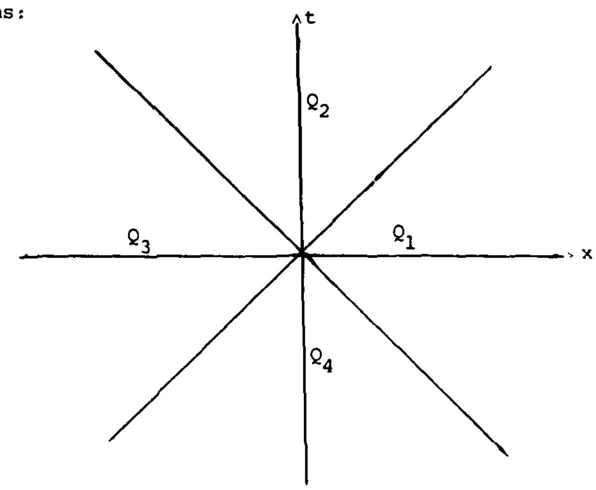
and, in particular,

$$(155) \qquad \left\{\begin{array}{l} u_\lambda(x,\tau(x)) = u(x,\tau(x)) = w(x,\tau(x)) \\ \dfrac{\partial u_\lambda}{\partial r}(r,\tau(x)) = \dfrac{\partial u}{\partial r}(x,\tau(x) - 0) = \dfrac{\partial w}{\partial t}(x,\tau(x)) \quad \text{a.e. on } \{x/|\tau'(x)| < 1\} \end{array}\right.$$

where we recall that τ , the line of influence, is Lipschitz continuous, with Lipschitz constant 1 .

We shall now use assumption (154) to obtain more information about the line of influence. We need the following notations:

$$(156) \qquad \begin{array}{l} Q_1 = \{(x,t)/x \geq |t|\} \\ Q_2 = \{(x,t)/t \geq |x|\} \\ Q_3 = \{(x,t)/x \leq -|t|\} \\ Q_4 = \{(x,t)/t \leq -|x|\} \end{array}$$



We shall denote

$$(157) \qquad \frac{\partial w}{\partial t}(x,t;Q_i) = \lim_{\substack{(h,k)\to 0 \\ (h,k)\in Q_i}} \frac{\partial w}{\partial t}(x+h,t+k) .$$

Thanks to (154), $\frac{\partial w}{\partial t}(x,t;Q_i)$ is defined for $1 \le i \le 4$, and we have the formula

$$(158) \qquad \frac{\partial w}{\partial t}(x,t;Q_1) = \frac{1}{\sqrt{2}}\left[\frac{\partial w^\ell}{\partial \xi}(x,t) + \frac{\partial w^r}{\partial \eta}(x,t)\right],$$

with notations (125). We have analogous formulae for the three other limits.

Lemma 19. Let x be such that $\tau'(x)$ is defined and $|\tau'(x)| < 1$. Suppose that

$$(159) \qquad \max_{1\le i\le 4} \frac{\partial w}{\partial t}(x,t;Q_i) < 0 .$$

Then there exists a neighborhood $(x-\varepsilon,x+\varepsilon)$ of x such that $|x'-x| < \varepsilon \Rightarrow \tau'(x')$ has left and right limits at every point and $|\tau'(x' \pm 0)| < 1$;

$$\sup_{1\le i\le 4} \frac{\partial w}{\partial t}(x',\tau(x');Q_i) \le -\ell < 0 .$$

Proof. The hypothesis (159) implies that, in a neighborhood N of $(x,\tau(x))$:

$$\sup_{1<i<4} \frac{\partial w}{\partial t}(x',t';Q_i) \le -\ell < 0 ,$$

therefore $w(x',\cdot)$ is strictly decreasing for x' close enough to x, and moreover if k is so chosen that $w_t(x,\tau(x)) + k\, w_x(x,\tau(x)) < 0$, then

$$w(x + kh,\tau(x) + h) < 0 .$$

Thus, there exists a unique solution to the problem:

$$(160) \qquad \begin{cases} w(x',\alpha'(x')) = 0 \\ \max(|x-x'|,|\sigma(x') - \tau(x)|) \le \alpha \text{ where } a \text{ is a small positive number.} \end{cases}$$

To prove that σ is identical to τ in an interval $[x-\alpha',x+\alpha']$ where α' may be smaller than α, we have to check that

$$|\sigma'(x')| < 1 \quad \text{a.e.} \quad \text{on} \quad [x-\alpha',x+\alpha'] .$$

The function σ is continuous indeed, as w is continuous and $t' = \sigma(x')$ is the unique solution of $w(x',t') = 0$ in N. We may not directly differentiate the relation $w(x',\sigma(x)')) = 0$, as we do not have the assumptions of the implicit function theorem. But, with the very same argument as in this theorem, and using notation (157), and its analogue for $\frac{\partial u}{\partial x}$, we have

$$(161) \qquad \left| \begin{array}{l} w(x'+h,\sigma(x+h)) = w(x,\sigma(x')) + w_x(x',\sigma(x');Q_1)\, h + \\ + w_t(x',\sigma(x');Q_2)(\sigma(x'+h) - \sigma(x')) + \varepsilon_1(|h| + |\sigma(x'+h) - \sigma(x')|) \\ \text{for all } h \text{ such that } (h,\sigma(x'+h) - \sigma(x')) \cdot Q_1 . \end{array} \right.$$

Here ε_1 is a function such that

$$\lim_{k\to 0} \frac{\varepsilon_1(k)}{k} = 0 .$$

By a standard argument

(162)
$$\lim_{\substack{h\to 0 \\ (h,\sigma(x'+h)-\sigma(x'))\in Q_1}} \left[\frac{\sigma(x'+h) - \sigma(x')}{h}\right] = -\frac{w_x(x',\sigma(x');Q_1)}{w_t(x',\sigma(x');Q_1)}$$

The same result holds in the three other quadrants Q_2, Q_3 and Q_4, and by choosing α' adequately small we shall have

$$\left|\frac{w_x(x',\sigma(x');Q_i)}{w_t(x'\sigma(x');Q_i)}\right| \leq 1-\varepsilon \quad \text{for} \quad |x-x'| \leq \alpha'$$

and thus

$(h,\sigma(x'+h) - \sigma(x')) \in Q_1$ for $h > 0$, small enough

$(h,\sigma(x'+h) - \sigma(x')) \in Q_3$ for $h < 0$, $|h|$ small enough

$|\sigma'(x')| \leq 1-\varepsilon$ a.e. on $[x-\alpha',x+\alpha']$

$\tau(x') = \sigma(x')$, and τ' has right and left limits at all points of $[x-\alpha',x+\alpha']$. ∎

Let us compare locally the solution of the linear Klein-Gordon equation (147) to the solution of an approaching problem with simpler initial data. Let

$$\tau(x_0) = t_0 , \qquad \tau_0'(x_0) = m = -\frac{w_x(x_0,t_0)}{w_t(x_0,t_0)} ,$$

$$\tau_0(x) = t_0 + m(x-x_0),$$

$$w_0(x,t) = w_t(x_0,t_0)(t-t_0) + w_x(x_0,t_0)(x-x_0),$$

$$u_0(x,\tau_0(x)) = w_0(x,\tau_0(x)) = 0$$

$$\frac{\partial u_0}{\partial t}(x,\tau_0(x)) = \frac{\partial w_0}{\partial t}(x,\tau_0(x)) = w_t(x_0,t_0)$$

$$S_0 = \{(x,t)/t > \tau_0(x)\}.$$

Then:

$$u_0(x,t) = \sqrt{(1-m^2)}\; w_t(x_0,t_0) \sin \frac{t-t_0-m(x-x_0)}{\sqrt{(1-m^2)}} .$$

With the help of (148), we have

$$
(163)\qquad \begin{cases} u(x,t) - u_0(x,t) = w(x,t) - w_0(x,t) - \\[2mm] \qquad - \dfrac{1}{2\lambda} \displaystyle\int_{T^-_{x,t}} J_0\left(\sqrt{\dfrac{(t-t')^2 - (x-x')^2}{\lambda}}\right)[(w \cdot 1_S)(x',t') - (w_0 \cdot 1_{S_0})(x',t')]dx'dt'. \end{cases}
$$

Let us estimate (163) for x and t such that

$$
(164)\qquad |x-x_0| + |t-t_0| \le C\sqrt{\lambda}
$$

and under the hypothesis that $|\tau'(x_0)| < 1$ and that $w_t(x_0,t_0)$ and $w_x(x_0,t_0)$ are well-defined. Then:

$$
|w(x,t) - w_0(x,t)| \le o(|x-x_0| + |t-t_0|) = o(\sqrt{\lambda}).
$$

To estimate the integral, let us first note that

$$
|w \cdot 1_S - w_0 \cdot 1_{S_0}| \le |w - w_0| \cdot 1_{S \cup S_0} .
$$

This relation comes from the fact that, locally, $w \cdot 1_{S'} = -w^-$ and $w_0 \cdot 1_{S_0} = -w_0^-$.

We define new variables X and T by

$$
t-t' = T\sqrt{\lambda}
$$

$$
x-x' = X\sqrt{\lambda} .
$$

Then the integral expression in (163) is estimated by:

$$
\lambda \int_{T^+_{0,0}} |J_0(\sqrt{T^2-X^2})|\,|w(x-X\sqrt{\lambda},t-T\sqrt{\lambda}) - w_0(x-X\sqrt{\lambda},t-T\sqrt{\lambda})|\,1_{S \cup S_0}(x-X\sqrt{\lambda},t-T\sqrt{\lambda})dXdT.
$$

But:

$$
|w(x-X\sqrt{\lambda},t-T\sqrt{\lambda}) - w_0(x-X\sqrt{\lambda},t-T\sqrt{\lambda})| \le o(|x-X\sqrt{\lambda}-x_0| + |(t-T\sqrt{\lambda}-t_0|)
$$

and we have to check that $\{(X,T) \in T^+_{00} / (x-X\sqrt{\lambda},t-T\sqrt{\lambda}) \in S \cup S_0\}$ is bounded. This set can be written as:

$$
\left\{ (X,T)/|X| \le T \le \frac{t}{\sqrt{\lambda}} - \min\left\{\frac{\tau(x-X\sqrt{\lambda}),\tau_0(x-X\sqrt{\lambda})}{\sqrt{\lambda}}\right\}\right\}
$$

and, using the fact that $|\tau'(x_0)| < 1$, this set is bounded, under the condition (164).

Thus, immediately:

$$
(165)\qquad |u(x,t) - u_0(x,t)| = o(\sqrt{\lambda}).
$$

A consequence of (165) is that, for λ sufficiently small, the solution u of (147) is negative on the set

$$
(166)\qquad T^-_{x_0,t_0 + (\cdot - \cdot)\sqrt{\lambda(1-m^2)}} \cap \{(x,t)/t \le \tau(x)\}.
$$

This uses the fact that $u_t < 0$ on a neighborhood of x_0, as was proved in Lemma 19.

Therefore, on the set (166), the solution of the penalized problem (122) is the solution of the linear problem (147), for λ small enough. We have thus, for (x,t) on the set (166):

$$\frac{\partial u_\lambda}{\partial t}(x',t') = \frac{\partial w}{\partial t}(x',t') - \frac{1}{2\lambda}\int_{(x',t-|x-x'|)\in S} \frac{\partial w}{\partial t}(x',t-|x-x'|)dx' -$$

$$- \frac{1}{2\lambda\sqrt{\lambda}}\int_{T^-_{x,t}} J_0'\left(\frac{\sqrt{(t-t')^2-(x-x')^2}}{\sqrt{\lambda}}\right)(t-t')(w\cdot 1_S)(x,t)dx'dt.$$

Reasoning as for (165), we can prove, under assumption (164) that

$$\left|\left(\frac{\partial u_\lambda}{\partial t} - \frac{\partial u_0}{\partial t}\right)(x,t)\right| = o(1),$$

or:

$$\left|\frac{\partial u_\lambda}{\partial t}(x,t) - w_t(x_0,t_0)\cos\frac{t-t_0-m(x-x_0)}{\sqrt{\lambda(1-m^2)}}\right| = o(1)$$

and, in particular

$$(167)\quad \begin{cases} \lim\limits_{\lambda\to 0} \dfrac{\partial u_\lambda}{\partial t}(x_0,t_0+(\pi-\varepsilon)\sqrt{\lambda(1-m^2)}) = +w_t(x_0,t_0)\cos(\pi-\varepsilon) \\[2ex] \text{on } \{x_0/\tau'(x_0) < 1 \text{ and } w_t(x_0,\tau(x_0);Q_i) < 0,\ i=1,\dots,4\}. \end{cases}$$

Analogously

$$(168)\quad \begin{cases} \lim\limits_{\lambda\to 0} \dfrac{\partial u_\lambda}{\partial x}(x_0,t_0+(\pi-\varepsilon)\sqrt{\lambda(1-m^2)}) = w_x(x_0,t_0)\cos(\pi-\varepsilon) \\[2ex] \text{on } \{x_0/\tau'(x_0) < 1 \text{ and } w_t(x_0,\tau(x_0);Q_i) < 0,\ i=1,\dots,4\}, \end{cases}$$

and (167) and (168) in turn imply:

$$(169)\quad \begin{cases} \lim\limits_{\lambda\to 0} \dfrac{\partial u_\lambda}{\partial \xi}(x_0,t_0+(\pi-\varepsilon)\sqrt{\lambda(1-m^2)}) = w_\xi(x_0,t_0)\cos(\pi-\varepsilon), \\[2ex] \lim\limits_{\lambda\to 0} \dfrac{\partial u_\lambda}{\partial \eta}(x_0,t_0+(\pi-\varepsilon)\sqrt{\lambda(1-m^2)}) = w_\eta(x_0,t_0)\cos(\pi-\varepsilon). \end{cases}$$

Therefore, the limit $\bar{u}$ of u satisfies:

$$\frac{\partial \bar{u}}{\partial t}(x,\tau(x)) = -\frac{\partial w}{\partial t}(x,\tau(x))$$

a.e. on $\{x/|\tau'(x)| < 1\}$.

This proves that $\bar{u}$ is indeed the solution of P_∞. ∎

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We give an explicit formula which describes the solution of the problem of the vibrating string with a plane obstacle. This formula allows us to prove continuous dependence on the data; a regularity result is given. We prove some results on the convergence of the penalized problem and give a numerical scheme.

A few results are given without the requirement that the obstacle be plane.

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